Inflationary Redistribution, Trading Opportunities and Consumption Inequality

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Inflationary Redistribution, Trading Opportunities and Consumption Inequality*

Timothy Kam[†] Junsang Lee[‡]

Abstract

We study *competitive search* in goods markets in a heterogeneous-agent monetary model. The model accounts for three stylized facts connecting inflation to consumption inequality, to price dispersion, and to the speed of monetary payments. With competitive search, individuals' endogenous probabilities on trading events give rise to a trading-opportunity (extensive-margin) force that works in opposite direction to well-known redistributive (intensive-margin) effect of inflation. This implies a new trade-off in response to long-run inflation targets. Welfare falls but liquid-wealth inequality falls and then rises with inflation as an extensive margin of trade dominates the redistributive intensive margin, when inflation is sufficiently high.

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Keywords: Competitive Search; Inflation; Policy Trade-offs; Redistribution; Computational Geometry.

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1 Introduction

After almost four decades of low—and in recent years very low—inflation, the U.S. inflation rate appears to be picking up again. During the Great Recession, inflation-targeting policy makers advocated policies that raised longer term inflation rates. Some policy makers in some countries still do. These policy responses motivate us to study the welfare and distributional consequences of inflation afresh. From a long-run perspective, should we be concerned about the effects of having higher or lower trend inflation? Are there possibly countervailing effects on household and business incentives, and on economic inequality, as a result of a particular long-run inflation policy?

In this paper, we focus on how might inflation contribute to inequality across heterogeneous individuals' consumption and monetary asset outcomes. We also study welfare costs of inflation in the long-run and along the transition between steady states, in terms of aggregate welfare and across heterogeneous individuals. We provide new insights on these matters from the perspective of a competitive-search model with heterogeneous agents. The model extends that of Menzio, Shi and Sun (2013).[1]

We show that this simple model—deliberately lacking in any exogenous, idiosyncratic shock assumptions—can account for three interesting micro- and macro-level facts.[2] The facts are as follows: (i) Inflation and consumption inequality have a hump-shaped relationship; (ii) inflation and (non-sales) goods pricing dispersion tend to be positively related; and (iii) inflation and the speed of consumer money payments are positively related.[3]

In addition, this alternative framework advances a new channel from inflation to asset and consumption inequality that is not present in existing heterogeneous-agent monetary models. The key feature in the model that induces this new channel is a well-known aspect of competitive search theory—an equilibrium trade-off between an *intensive margin* of consumption and an *extensive margin* in terms of trading probabilities. Since the model also features a non-degenerate distribution of agents, such a trade-off would also be het-

[1]The flavor of competitive search markets we have in mind is the same as that in Moen (1997) or Menzio et al. (2013): In such markets, trade is decentralized in that there is no single Walrasian or centralized market. Excess demand in Walrasian markets is eliminated by some unspecified invisible-hand pricing outcome. In competitive search, agents have to direct their search to observed trading posts (viz. terms of trade) created by sellers. Sellers anticipate buyers' search strategies and post optimally. An equilibrium is given by a distribution of trading posts or market tightness measures (or equivalently, a distribution of terms of trade) that is consistent with optimal buyer search, firm posting, consumption and production behavior.

[2]In this paper, our aim is to quantitatively discipline a microfounded model of money with heterogeneous agents and to show that it possesses some realism while providing a new economic insight or trade-off that is missing in more reduced-form models of money and agent heterogeneity. As such, it would be counterproductive to understanding if we added more exogenous sources of micro-level heterogeneity in order to "match" empirical facts.

[3]We expand on these facts further below, in Section 1.2. Why the focus on consumption inequality? Consumption is vital in inequality measurements since almost every economic model has a basic utility function that depends on it. Furthermore, paying attention to consumption inequality allows one to see directly whether there are impediments to consumption smoothing (Attanasio and Pistaferri, 2016).

erogeneously dependent on individual states. We show that through this new channel, a higher long-run inflation policy tends to attenuate (*exacerbate*) wealth inequality but increase (*reduce*) consumption inequality for sufficient low (*high*) range of inflation. As a corrollary, we also show that the welfare cost of inflation is nontrivial, especially when transitional dynamics are taken into account.

Inflation in the model induces a redistributive-tax effect through the intensive margin of trade in all markets. This is also the case in standard heterogeneous-agent monetary models. However, inflation also raises individuals' downside risks or probabilities of not getting to trade in decentralized search markets. This risk affects individual incentives—i.e., how fast agents trade relative to how much liquidity they carry into in these markets. In equilibrium, agents have to trade off between these two channels, and the trade-off depends on long-run inflation targeting policy.

1.1 The literature and what we do

Consider a taxonomy of the costs and benefits of inflation from the perspective of standard Walrasian-market models (see, e.g., Erosa and Ventura, 2002). First, inflation acts as an intertemporal tax that distorts consumption. This feature raises the (welfare) cost of inflation present in all monetary models (with or without heterogeneous agents). Second, inflation is costly since agents have to engage in precautionary liquidity management activities. Third, inflation may act as a redistributive tax that shifts resources from the "rich" to the "poor". This force tends to lower the welfare cost of inflation.

In most heterogeneous-agent models (see, e.g., Imrohoroğlu and Prescott, 1991a; Akyol, 2004; Boel and Camera, 2009; Meh et al., 2010), the third force—a redistributive-tax channel of inflation—is strong. This is often because there is only an intensive margin through which inflation tax works. That is, with increasing inflation, agents would like to reduce their money holdings. Those with high balances reduce their holdings more relative to those at the bottom end of the distribution. This tends to lower average money balance. Hence, inflation acts as a progressive tax that reduces inequality of money holdings. Consequently, agents' decision margins—and the impact of government policy on private decisions—typically work along the intensive margin.[4] This explains why in many heterogeneous-agent models, the welfare cost of inflation is often smaller than representative-agent models (Camera and Chien, 2014).

However this conclusion need not be robust, as it may depend on the nature of idiosyncratic shocks, financial structure and the sensitivity of labor supply to real wage changes: Camera and Chien (2014). For example, in their model with a reduced-form

[4]One can also complicate Walrasian models with extensive margins to capture limited market participation, as in Alvarez et al. (2002). However, conditional on being in any market, agents within those markets are always trading. In competitive search, some agents are participating and searching but may not find a match in order to trade.

transaction technology that is free from inflationary effects, Erosa and Ventura (2002) showed that for sufficiently high returns to scale of the technology, inflation can become a regressive tax. In a different Bewley economy, where holding money is driven by precautionary motives, Wen (2015) showed that inflation can increase agents' consumption risks by tightening poorer agents' ad-hoc borrowing limits. There is also another class of models that combine these incomplete-markets and ex-post heterogeneity features with sticky price assumptions (see, e.g., Kaplan, Moll and Violante, 2018; Ravn and Sterk, 2020).

In a random-matching, search-theoretic model of money, Molico (2006) showed that a "real balance effect"—i.e., agents choosing to carry less money balances into decentralized trades as inflation increases which results in a higher amount of money paid per goods—can work against the redistributive effect of inflation. There is a similar "reduced-form" property in our model, but with a deeper twist. In our setting with competitive search in decentralized trades, this is bolstered by the additional extensive margin effect: Higher inflation exacts a greater downsize risk of not matching for agents by reducing the equilibrium matching probability for buyers (*contra*. random matching). Although expected money carried in each decentralized trade will be lower per payment for goods, with lower equilibrium probability of matching, agents who match don't have to reduce consumption as much—this trade-off between matching probability and quantity of goods in the competitive search environment (see, e.g., Peters, 1984, 1991; Moen, 1997; Burdett et al., 2001; Julien et al., 2008; Shi, 2008) helps to amplify the speed at which agents expect to get rid of their money in decentralized trades.

Chiu and Molico (2010) also have a notion of extensive margin, in the form of costly participation in centralized markets. In our setting, even without costly participation in markets, there is a non-trivial extensive margin. In Chiu and Molico (2010) and Rocheteau, Weill and Wong (2019), trading probabilities are fixed in decentralized-market meetings. This is due to their random matching assumption. In our setting, the extensive margin arises in the form of endogenous matching probabilities.

In contrast to the Walrasian or random matching models discussed above, our Menzio, Shi and Sun (2013) competitive search setup advances a fourth channel to the standard taxonomy previously outlined.[5] Our framework is closer to the monetary search literature in which money as a medium of exchange and its liquidity properties are explicitly modelled. In our setting, there is an opposing extensive margin effect that helps to mitigate the previously-discussed redistributive channel of inflation. With higher inflation, agents are also spending faster in decentralized trades and entering the Walrasian or cen-

[5] Another advantage of considering competitive search is that agents' decision problems are block recursive (as pointed out earlier in Menzio et al., 2013). This means that agents' decision problems are recursively independent from the problem of determining the equilibrium distribution of assets. This has an accuracy benefit in terms practical computation—one does not have to *ad-hoc* parametrize aggregate wealth distribution as state variables when computing transitional dynamics.

tralized market to rebalance their liquidity more frequently. Higher centralized-market participation implies that there are more agents at the bottom end of the equilibrium distribution at the end of each period. These are agents who will enter the centralized market in the subsequent period. Also, there are less agents at the upper end of the distribution since they top up with less liquidity in the centralized market and they spend faster in the decentralized search market. Since agents have a preference to consume goods from both types of markets, we also show that this qualitative equilibrium behavior is robust to whether there is (or there is not) additional friction to centralized market participation. The additional friction is parametrized by a fixed pecuniary cost of centralized-market participation, similar in spirit to Chiu and Molico (2010).

This will be key to inducing non-monotone effects of inflation on consumption and money inequality. Of course, endogenous matching probabilities via competitive search is not new (see, e.g., Rocheteau and Wright, 2005). What is different here is that the endogenous matching probabilities are dependent on heterogeneous agent states. This creates a nontrivial equilibrium, countervailing effect to what would be a traditional redistributive role of inflation. This is an important feature driving our non-monotone inequality results.[6]

Sun and Zhou (2018) study fiscal and monetary policy in a similar setup. However, they assume that all agents in a decentralized market must exit it in one period and must enter a centralized market deterministically. In our setting, agents get to choose. In their equilibrium, since agents have quasilinear preferences in the centralized market, agent heterogeneity (i.e., non-degeneracy of the equilibrium distribution of agents) needs to be preserved by assuming that there are exogenous idiosyncratic shocks to agents in the centralized market. In contrast, we do not require additional exogenous individual shocks. We can have a non-degenerate distribution of agents in our model since agents endogenously choose which market to enter every period and not all agents go to the centralized market at the same time.

[6] An expanded version of our model can be shown to relate to two elegant intellectual origins. In the Online Appendix A, we entertain the additional feature of an exogenous probability α that ex-ante, each agent may go to the CM costlessly. In that extended setting, we can relate to two well-known models in the literature—i.e., the representative-agent random-matching model with competitive search DM of Rocheteau and Wright (2005), and, the block-recursive ex-post heterogeneous agent model of Menzio et al. (2013). The main difference in one limit of our model to Rocheteau and Wright (2005) is that in Rocheteau and Wright (2005), some measure of households become sellers in the DM each period. In our setting, non-DM-buyer households are, in a sense, sellers only insofar as supplying labor to firms that create trading posts in the DM. This is a feature inherited from Menzio et al. (2013). Our difference to Menzio et al. (2013) is that here, agents derive consumption value in the CM and they need not exhaust all their liquid wealth before deciding to enter the CM again. We extend the theoretical analyses of Menzio et al. (2013) in the direction of inflationary monetary equilibria and prove their existence. We also provide an efficient computational method for solving these models, thus taking the new monetarist literature closer to mainstream quantitative macroeconomics.

1.2 Three motivating empirical facts

Previously, we alluded to three empirical facts. We now discuss these facts in more detail.

Fact 1 (Hump-shaped inflation-consumption-inequality relation). In Figure 1, we reproduce the U.S. consumption inequality measure (standard deviation of log consumption) from Attanasio and Pistaferri (2016), and we juxtapose this against CPI inflation data obtained from FRED.[7] Inflation has been trending downwards since the early 1980s Volcker era. Consumption inequality between 1980 and 2005 has been rising. However, it has been falling since 2005.

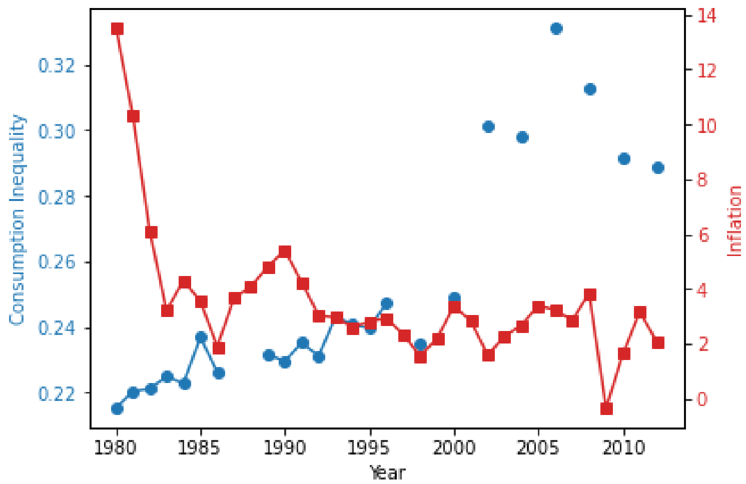

Figure 1: Inflation (*right y-axis*) and consumption inequality (*left y-axis*).

In other words, from 1980 to 2004, consumption inequality and inflation were negatively correlated but this correlation became positive from 2005 onwards. The correlation charts for this relationship pre- and post-2005 are in Figure 2 (respectively, the *left* and *right panel*). That is, there is a *hump-shaped* relationship between inflation and consumption inequality over a relatively long-run span of time.

This hump-shaped relationship would still hold if we considered alternative measures of consumption inequality. Meyer and Sullivan (2017) used the ratio of consumption outcomes for top percentiles relative to bottom percentiles (e.g., the ubiquitous "90/10" ratio) to measure inequality in the U.S. consumption distribution since the 1960s. They also find that since 2005 consumption inequality has been falling despite rising income inequality.[8] If we superimpose the observation of Meyer and Sullivan (2017) against

[7]See FRED, series FPCPITOTLZGUSA.

[8]The figures are reproducible from our Jupyter notebook `consumption-inequality-inflation.ipynb` (available from our Github repository).

inflation, we would again see the same hump-shaped relation between inflation and consumption inequality as in Figures 1 and 2.[9]

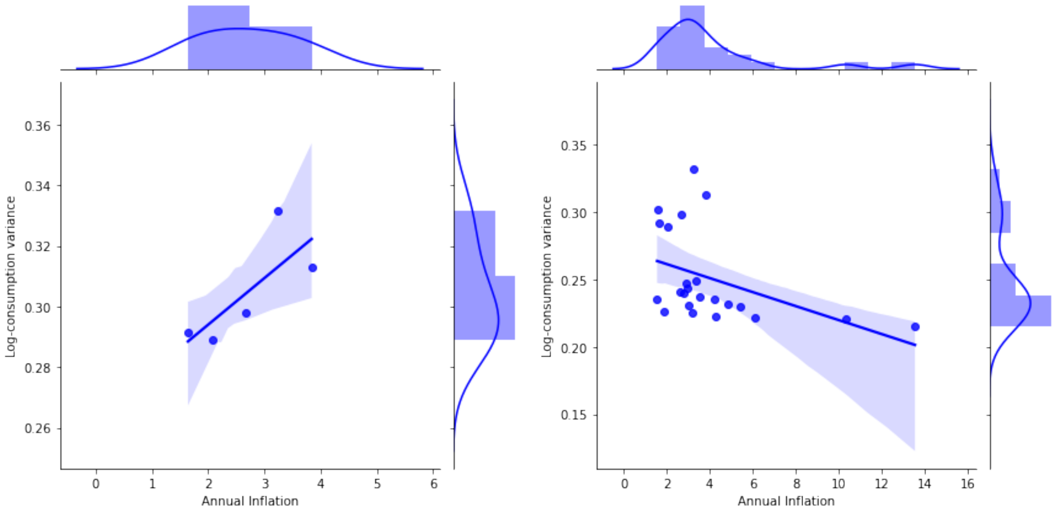

Figure 2: Hump-shaped inflation-consumption-inequality relation. *Left*: After (2005-2012). *Right*: Before (1980-2004). *Notes*: Least-squares regression (solid line), bootstrapped 95% confidence bands (shaded patches), and marginal data histograms (smoothed density estimates) superimposed.

Fact 2 (Inflation and price dispersion). In a recent paper, Sheremirov (2020) documented the following facts about inflation and price dispersion of goods at the Universal Product Code (UPC) level, using IRI scanner data (a proprietary data set): The comovement between dispersion of regular retail prices and inflation is positive in the data. In studying city-level data from the U.S., Debelle and Lamont (1997) showed that price dispersion and inflation are positively associated.[10]

Fact 3 (Inflation and speed of money transactions). Consider a measure of how quickly people spend money. The data comes from the U.S. Federal Reserve Bank of Atlanta's *Survey of Consumer Payment Choice* (SCPC) and *Diary of Consumer Payment Choice* (DCPC). The data is available on an annual frequency. The SCPC measures how often a means of

[9] A similar phenomenon has been documented for the UK and Sweden. Blundell and Etheridge (2010) document that consumption inequality measures in the UK have been increasingly divergent from other measures based on income and earnings. Daunfeldt et al. (2010) document a similar divergence for Sweden since 1988. Inflation in both the UK and Sweden have also been on a downward trend since the late 1980s.

[10] Reinsdorf (1994) uses micro-level data from the BLS for the years 1980 to 1982 and found that this relationship is negative. Sheremirov (2020) also found that the comovement between dispersion of *sales* prices and inflation is *negative* in the data. Since our model will be silent on the separate phenomenon of *sales* and *sale prices*, we will focus on the majority of evidence on a positive relationship between retail prices and inflation.

payment (e.g., cash or credit card) is used in consumer expenditures. The DCPC measures the dollar value of expenditures, conditioning on each payment instrument. The data on this phenomenon is still somewhat sparse. The SCPC is available from 2008 and the DCPC is available from 2015. Nevertheless, we will take the information below to be a suggestive fact.

We use the SCPC data on the number of transactions using cash (b) and the DCPC for the size of expenditure for each cash transaction (x). We also use the aggregate M1 series (M). We define the speed of money transactions here as the mean of $(b \times x)/M$. This can be interpreted as how quickly agents expend their money holdings on average. This measure will also be consistent with our model's measure of money-transaction speed. However, we need to take a stance on what is money, in the payment instruments measures. There are nine types of payments in the dataset: Cash, check, money order, debit card, credit card, prepaid card, bank-account-number payment, online-banking-bill payment and direct-from-income payment. In our online notes, we consider the following measurement cases: (1) cash; (2) case (1) + check + money order + debit card; (3) case (2) + credit card + prepaid card; (4) case (3) + bank-account-number payment + online-banking-bill payment; and (5) case (4) − credit card.[11] (The "addition" and "subtraction" here denote dataset-wise inclusion and exclusion, respectively.)

[11] See the Jupyter notebook `payments-speed-inflation.ipynb` on our Github repository.

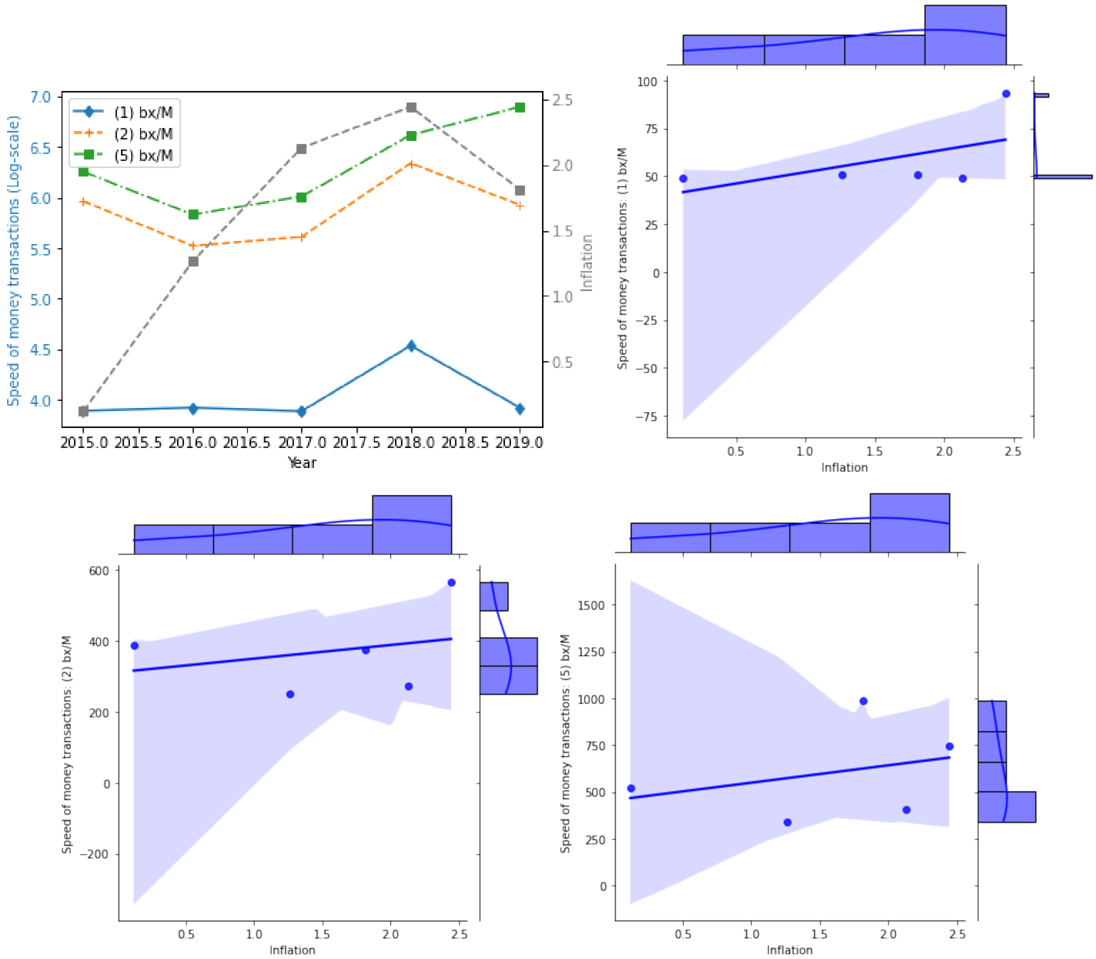

Figure 3: Speed of money payments (bx/M measures 1, 2 and 5) and inflation. *Top-Left*: Time series with the right-hand-side axis measuring inflation (dashed-square marker). *Top-Right*: Scatterplot of bx/M (measure 1) and inflation. *Bottom-Left*: Scatterplot of bx/M (measure 2) and inflation. *Bottom-Right*: Scatterplot of bx/M (measure 5) and inflation. *Notes*: Least-squares regression (solid line), bootstrapped 95% confidence bands (shaded patches), and marginal data histograms (smoothed density estimates) superimposed onto scatterplot (*right*).

Consider the most relevant measures of money payments, cases (1), (2) and (5). We plot the available time-series observations of these against that of CPI inflation in Figure 3 (*top-left panel*). In Figure 3 (*top-right panel*), we see that the speed at which agents spends using *cash* only—i.e., case (1)—appears to be positively correlated with inflation. That is, in a more inflationary environment agents are expected to be quicker in spending their money holdings on goods. As additional checks, we also show that a similar pattern holds when we consider a broader measure of money payments—i.e., cases (2) and (5), respectively, in Figure 3 (*bottom-left panel*) and Figure 3 (*bottom-right panel*).[12] This is quite

[12]In fact, the pattern still holds in cases (3) and (4) as well. We prefer not to use these cases since they include credit cards. (Credit-card payments could be largely dominated by consumers taking out debt as opposed to just being a means of money transfer.)

intuitive and will be a feature implied by our model's equilibrium.

The remainder of this paper is organized as follows. In Section 2, we set up and analyze a version of the model of Menzio, Shi and Sun (2013) in a more general setting with non-zero inflation. In Section 3, we take the model to the data. In Section 4, we provide an explanation of the main equilibrium mechanism or trade-offs that drive the model's quantitative and welfare outcomes. Armed with these insights, we conduct counterfactual experiments in Section 5 to understand how these forces balance out in monetary equilibria under alternative long-run inflation targeting policies. We will reconcile and rationalize these three motivating facts from the lens of a novel monetary model with decentralized, competitive search markets. Here, we also study their welfare and monetary-wealth inequality consequences. Finally, we also show that our insights are robust to a limiting case where we shut down an additional friction due a limited-CM-participation assumption. We conclude with Section 6.

2 Model environment

There is a decentralized market (DM) with competitive search and matching friction and a centralized market (CM). The DM allows one to microfound the crucial frictions in the model. The CM, as in Lagos and Wright (2005), allows the model to be more flexibly mapped to macroeconomic data and to be compared with more standard models with neoclassical origins. Time is discrete and indexed by $t \in \mathbb{N}$. Hereinafter, we will denote $X := X_t$ and $X_{+1} := X_{t+1}$ for dynamic variables.

Market incompleteness will arise from two features of the model: First, equilibrium matching in the DM (where money is essential) implies that agents face ex-ante uncertainty over being able to exchange and consume in those markets. Since agents are "anonymous" in these markets, their individual trading risks cannot be insured away by exchanging private state-contingent securities. As a result, we have ex-post agent heterogeneity. Second, because agents have a convex preference over goods from the centralized and the decentralized markets, there is endogenous limited participation in centralized markets. Agents engage in the centralized markets for liquidity risk management. For quantitative purposes we introduce a couple of modifications to Menzio et al. (2013): a preference for consumption in the CM and a possibly non-zero fixed cost of CM participation. Later, we show that these do not affect the qualitative insights of the model.

Figure 4, which summarizes the timing of events and decisions between two arbitrary dates, may be a useful accompaniment to the notation and decision processes to be described next.

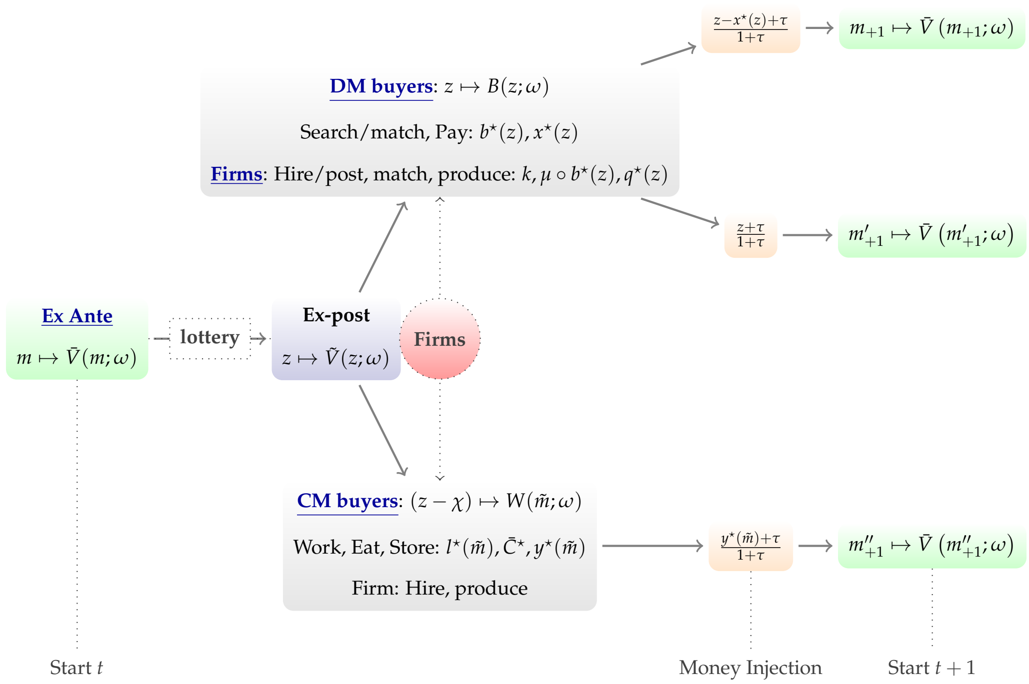

Figure 4: Timing, Markets, Outcomes

2.1 Money supply

We assume that the total stock of money in the economy M grows according to the process

$$\frac{M_{+1}}{M} = 1 + \tau, \tag{2.1}$$

where $\tau > \beta - 1$, the growth rate of money supply, is a given policy parameter.

Following Menzio et al. (2013), we define labor as the numéraire good. If we denote ωM as the current nominal wage rate, where ω is normalized nominal wage (i.e., nominal wage rate per units of M), then a dollar's worth of money is equivalent to $1/\omega M$ units of labor. The variable ω will be endogenously determined in a monetary equilibrium.[13] If M is the beginning of period aggregate stock of money in circulation, then $1/\omega = M \times 1/\omega M$ is the beginning of period real aggregate (per-capita) stock of money, measured in units of labor.

Denote (equilibrium) nominal wage growth as $\gamma(\tau) \equiv \omega_{+1} M_{+1}/(\omega M)$. Later, for a

[13]In the original setting of Menzio et al. (2013) the unique good traded in the perfectly-competitive Walrasian spot market is labor. Hence the authors decision to define labor as the numéraire. We maintain their definition for ease of comparison. For readers who prefer the more conventional normalization of a Walrasian consumption good as the numéraire, it will just be a matter of making an appropriate relative price conversion—i.e., multiply the quantity of a particular good (measured in labor units), say q, by the relative price $\omega M/p$, where p is some nominal price level of another (Walrasian) good.

stationary monetary equilibrium, we will require that equilibrium nominal wage grows at the same rate as money supply, i.e., $\gamma(\tau)|_{(\omega_{+1}=\omega)} = M_{+1}/M$.

2.2 Markets, agents, commodities and information

There is a measure one of individuals who decide at the beginning of each date which market (CM or DM) to participate in. Firms act in both CM and DM at the same time. An individual can only be in the CM or DM at a given time period. In the DM, individuals shop for special goods q. In the CM individuals supply labor l, and, consume a general good C. As a result, they also manage their liquidity holding y to be carried into the following period. A firm in the CM hires labor to produce the general CM good and the special DM goods. We describe the CM and DM markets in turn.

In the CM, two markets are open: A competitive spot market for labor and a competitive general good market. Agents demand money as a precaution against the need for liquidity in anonymous markets in the DM.

In the DM, we have a setting similar to Menzio et al. (2013) where there is an information friction: Buyers of special DM goods, q, are anonymous and cannot trade using private claims or cannot undertake contracts with selling firms. As a result, the only medium of exchange is money. There is a finite set of *types* of individuals and goods, I. There is a continuum of individuals and firms of type $i \in I$, where an individual i consumes good i and produces good $i+1$ (mod-$|I|$). A type i firm hires labor service from type $i-1$ (mod-$|I|$) individuals (from the CM spot labor market) and transforms it (linearly) into the same amount of DM good i. Each i-type firm commits to posted terms of trade in all submarkets it chooses to enter. Buyers of good i direct their search toward these submarkets that sell good i, by choosing the best terms of trade offered. However, as we will see, these buyers will have to balance their decision on terms of trades against the probability of getting matched. Since firms and buyers choose which submarket to participate in, a type i buyer will only participate in the submarkets where type i firms sell.

There is a continuum of submarkets for these types of goods. Each type-i good is equivalently indexed by a submarket's terms of trade $(x, q) \in \mathbb{R}^2_+$. A submarket terms of trade comprises a real payment by a buyer, x, and the quantity traded in exchange, q. Hereinafter, the explicit dependency on the type of good $i \in I$ will become unnecessary.

2.2.1 Preference representation

The per-period utility function of an individual is

$$U(C) - h(l) + u(q). \tag{2.2}$$

We assume that the functions U and u are continuously differentiable, strictly increasing, strictly concave, $U_1, u_1 > 0$, $U_{11}, u_{11} < 0$, and the following boundary conditions hold: $u(0) = U_1(\infty) = u_1(\infty) = 0$, and $u_1(0) < \infty$.[14] Also, we assume that $h(l) = l$. This simplifies the algebraic description of the CM decision problem and ensures that agents exiting the CM are identical.

2.2.2 Matching technology in the DM

We follow the assumptions of Menzio et al. (2013) in the setting below. Let $\theta \in \mathbb{R}_+$ denote the ratio of trading posts to buyers in a submarket—i.e., its market tightness. In a submarket with tightness θ, the probability that a buyer is matched with a trading post is $b = \lambda(\theta)$. The probability a trading post is matched with a buyer is $s = \rho(\theta) := \lambda(\theta)/\theta$. We assume that the function $\lambda : \mathbb{R}_+ \to [0, 1]$ is strictly increasing, with $\lambda(0) = 0$, and $\lambda(\infty) = 1$. The function $\rho(\theta)$ is strictly decreasing, with $\rho(0) = 1$, and $\rho(\infty) = 0$. We can re-write a trading post's matching probability $s = \rho(\theta) = \rho \circ \lambda^{-1}(b) \equiv \mu(b)$. Observe that the matching function μ is a decreasing function, and that $\mu(0) = 1$ and $\mu(1) = 0$. Assume that $1/\mu(b)$ is strictly convex in b.

2.2.3 Firms

Consider a firm $i \in [0, 1]$ that takes the CM good's relative price p (in units of labor) as given. The firm hires labor on the spot market and transforms hired labor services into Y units of CM good linearly. In the DM, a firm takes the market tightness function θ as given, and chooses the measure of trading posts (viz., shops) $\mathrm{d}N(x, q)$ to open in each submarket. (This is equivalent to stating that the firms post and commit to their terms of trade in the particular submarket, taking the probability of being matched with a buyer as given. If x is what a matched buyer is willing to pay for q and $s(x, q) := \rho(\theta(x, q))$, then $x \cdot s(x, q)$ is the firm's expected revenue in submarket (x, q). To produce q the firm must hire $c(q)$ units of labor. Hence $s(x, q)c(q)$ is its expected labor wage bill at submarket (x, q). We assume that $q \mapsto c(q)$ is a continuous convex function. The firm also pays a per-period fixed cost k of creating the trading post in submarket (x, q).

The firm's profit is:

$$\pi(p; k) = \max_{Y \in \mathbb{R}_+} \{pY - Y\} + \max_{\mathrm{d}N} \int_{\mathbb{R}^2_+} \{s(x, q) [x - c(q)] - k\} \, \mathrm{d}N(x, q), \tag{2.3}$$

where N is a positive measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^2_+)$. The first term on the RHS

[14] We will use the notational convention, $f_i(x_1, ..., x_n) \equiv \partial f(x_1, ..., x_n)/\partial x_i$, to denote the value of the partial derivative of a function f with respect to its i-th variable. Likewise, f_{ij} will denote its cross-partial derivative function with respect to the j-th variable.

is the firm's value from operating in the CM. The second, is its DM total expected value across all submarkets it chooses to operate in.

Note that the firms' problem above (and also agent decision problems to be discussed below) do not explicitly depend on the aggregate distribution of agents. This is because of the nature of competitive search in the DM: Firms and buyers take matching probabilities as given when making respective posting and directed search decisions. The observed terms of trade posts and matching probabilities suffice to condition their decision processes. Moreover CM preferences are quasilinear such that agents are identical at the end of the CM. We discuss this further in Section 2.4.1.

2.3 Individuals' decisions

An individual is identified by her current money balance (measured in units of labor), m. Given policy τ, her decisions also depend on knowing the aggregate wage ω. Denote the relevant state vector as $\mathbf{s} := (m, \omega)$.[15] At the beginning of a period (ex ante), an individual decides whether to work and consume in the CM or whether to be a buyer in the frictional DM. Ex post, if the agent has positive initial money balance as a DM buyer, he continues searching for a trading post. Also, ex post, another agent is in the CM either because she had previously expended all her money in a DM submarket or she finds it optimal to go to CM even with positive money balance.[16] Next, we describe these different ex-post agents' problems in turn, and then, we will describe an agent's ex-ante decision problem.

2.3.1 Ex-post individual in the CM

Suppose now we have an individual $\mathbf{s} := (m, \omega)$ who begins the current period in the CM. The individual takes policy, τ, and the sequence of aggregate prices, $(\omega, \omega_{+1}, ...)$, as given. Her value from optimally consuming C, supplying labor l, and accumulating end-of-period money balance y, is

[15] In a steady state equilibrium, ω is a constant.

[16] Our assumption here is different to that of Sun and Zhou (2018). In Sun and Zhou (2018), all DM individuals go into the CM deterministically at the end of one period. From CM, individuals choose whether to go into the DM submarkets. In their model, at the end of every period agents would be identical since in the CM agent preferences are quasilinear. To avoid degeneracy in the agent distribution Sun and Zhou (2018) introduce Bewley-style idiosyncratic shocks. They do so in terms of preference shocks—i.e., labor supply shocks. In contrast, our model still preserves non-degeneracy without an additional assumption of exogenous preference shocks, since there is always a positive measure of agents who will be stuck trading in the DM submarkets for some time before some of them get to go to the CM.

$$W(\mathbf{s}) = \max_{(C,l,y)\in\mathbb{R}^3_+} \left\{ U(C) - h(l) + \beta \bar{V}(\mathbf{s}_{+1}) : pC + y \leq m + l,\ m_{+1} = \frac{\omega y + \tau}{\omega_{+1}(1+\tau)} \right\}, \tag{2.4}$$

where $\bar{V} : S \to \mathbb{R}$ is her continuation value function, to be fully described in Section 2.3.3 on the following page. This continuation value function yields her next-period expected total payoff from state m_{+1} . The continuation state for the individual, m_{+1}, is derived as follows: At the end of the CM, the individual would have accumulated balance y (measured in units of labor). In current units of nominal money, this is $\omega M \times y$. At the beginning of next period, each individual gets a nominal transfer of new money τM (population is normalized to size 1). In units of labor next period, the beginning-of-period balance would thus be $m_{+1} = (\omega M y + \tau M) / (\omega_{+1} M_{+1})$. Replacing for M/M_{+1} with the money supply process in (2.1) gives the expression for the individual's continuation state m_{+1} in (2.4).

2.3.2 Ex-post individual buyer in the DM

Now we focus on an individual who has just decided to be a DM buyer. The buyer chooses which submarket (or trading post) (x, q) to enter, taking the market tightness function $(x, q) \mapsto \theta(x, q)$ as given. The individual buyer, $\mathbf{s} := (m, \omega)$, has initial value:[17]

$$\begin{aligned} B(\mathbf{s}) = \max_{x\in[0,m], q\in\mathbb{R}_+} \Bigg\{ & \beta \left[1 - b(x,q)\right] \left[\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1} \right) \right] \\ & + b(x,q) \left[u(q) + \beta \bar{V}\left(\frac{\omega(m-x) + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1} \right) \right] \Bigg\}. \end{aligned} \tag{2.5}$$

Consider the first two terms on the RHS of the functional (2.5): With probability $1 - b(x,q) := 1 - \lambda(\theta(x,q))$ the buyer fails to match with the trading post and must thus continue the next period with his initial money balance subject to inflationary transfer. With the complementary probability $b(x,q) := \lambda(\theta(x,q))$ he matches with a trading post (x,q), pays the seller x in exchange for a flow payoff $u(q)$, and then continues into the next period with his net balance, also subject to inflationary transfers.

[17] Implicit in the DM-buyer's problem here is that the buyer sees which trading posts—each indexed by its posted terms of trade (x,q)—are open. In equilibrium, each buyer's problem must be consistent with firms' price-posting strategies (to be further discussed in Section 2.4).

2.3.3 Ex ante decision

Given a money balance z, the individual decides which markets to participate in, and her value becomes

$$\tilde{V}(z,\omega) = \begin{cases} \max_{a\in\{0,1\}} \{aW(z-\chi,\omega) + (1-a)B(z,\omega)\}, & z-\chi \geq -y_{\max}(\omega;\tau) \\ B(z,\omega), & \text{otherwise} \end{cases}, \tag{2.6}$$

where

$$y_{\max}(\omega;\tau) := \bar{m} - \frac{\tau}{\omega}, \tag{2.7}$$

is a natural upper bound on CM saving (in real money balances) and $\bar{m} \in (0,\infty)$ is a sufficiently large constant. We derive this upper bound in the Online Appendix B.

As shown in Menzio et al. (2013), the resulting value function B in Equation (2.5) may not be strictly concave in m.[18] This is the case even if primitive functions are. As a result, the value function $\tilde{V}$ may not be concave either.[19] This implies that agents can be weakly better off by choosing a lottery over the pure participation outcomes. That is, consider the following problem. Suppose at the beginning of a period, an agent begins with money balance m. If there is a non-empty subset $[z_1, z_2]$ containing m such that any weighted average of the pure-action induced values $\tilde{V}(z_1,\omega)$ and $\tilde{V}(z_2,\omega)$ (weakly) dominates $\tilde{V}(m,\omega)$, then the agent will optimally play a fair lottery $(\pi_1, 1-\pi_1)$ over the prizes $\{z_1, z_2\}$. This yields the *ex-ante* value

$$\bar{V}(\mathbf{s}) = \max_{\pi_1\in[0,1],z_1,z_2} \left\{\pi_1\tilde{V}(z_1,\omega) + (1-\pi_1)\tilde{V}(z_2,\omega) : \pi_1 z_1 + (1-\pi_1)z_2 = m\right\}. \tag{2.8}$$

Observe that in Equation (2.6), contingent on realizing a lottery payoff z, the outcome of the lottery also induces the pure action of going to the DM or the CM. If the agent decides to go to the CM, he must pay a (small) fixed cost $\chi \geq 0$ (measured in units of labor) to participate in the CM. This fixed-cost component is interpretable as a barrier to participation in liquidity-risk management in the CM for some agents.

Remark 1. This additional friction, parametrized by χ, is not *per se* crucial to the model's equilibrium mechanism and trade-off. In the quantitative model later, we allow for $\chi > 0$ in order to better fit auxiliary empirical targets. Theoretically, the extensive-margin CM-

[18]This is due to the bilinear and non-concave interaction between $b(x,q)$ and $u(q)$ in the DM-buyer's objective function in Equation (2.5). These two terms, respectively, give rise to an extensive margin (i.e., how likely is a buyer to trade) and an intensive margin (i.e., how much of q to consume given a match).

[19]However, we show that B it is still continuous and increasing in Online Appendix D (Lemma 5). Hence $\tilde{V}$ also inherits this property.

participation decision is still present even if $\chi = 0$. Why? From Equation (2.2) the flow preference function is strictly concave, implying that they would like to consume both CM and DM goods in their infinite lifetimes. However, no agent in equilibrium would optimally stay in either market forever. Agents would still go from CM to DM since they have a preference for DM goods too. Agents in DM would exit to CM at some optimal time since they need to manage their liquidity balance for future DM expenditures. In our quantitive experiments, we confirm this via a robustness check for the case where $\chi = 0$ (see our Online Appendix I.2).

In the ex-ante market participation problem (2.6), there is a limited short-sale (I.O.U.) constraint $z - \chi \geq -y_{\max}(\omega; \tau)$. It may be possible that an agent, whose state is such that $m < \chi$, when faced with deciding to go to the CM, may still find it optimal to issue an I.O.U. worth $m - \chi$ at the beginning of a CM, and go to work in the CM immediately to repay the shortfall $m - \chi$. Since the fixed cost is levied in the CM, and in the CM promises or contracts are completely sustainable, then a limited amount of short selling (I.O.U.) is possible. The limit on the short sale $m - \chi$, is equivalent to agents exerting a maximal CM labor effort $l_{\max}(\omega; \tau) = y_{\max}(\omega; \tau) + U^{-1}(1) < 2U^{-1}(1)$, and not saving anything in the CM. In Online Appendix C we derive these limits of $-y_{\max}(\omega; \tau)$ and $l_{\max}(\omega; \tau)$.

2.4 Monetary equilibrium

Clearly there exists a non-monetary equilibrium whereby no agent will participate in the DM. In this paper, we restrict attention to the case of a monetary equilibrium. Hereinafter, whenever we refer to "monetary equilibrium", or "equilibrium", we mean a recursive monetary equilibrium—one in which agent's decision functions are recursive and time-invariant maps. In what follows, we first characterize the equilibrium strategy of firms (section 2.4.1), the equilibrium value and decision functions of agents in the CM (section 2.4.2) and in the DM (section 2.4.3), and then close the equilibrium notion by describing the market clearing conditions (section 2.4.4). At the end of this section, we restrict attention to and define formally the notion of a stationary monetary equilibrium (SME).

2.4.1 Equilibrium strategy of firms

A firm's problem is static. We can characterize the equilibrium behavior of a firm given p (in the CM). Free entry in the CM will render zero profits to firms in equilibrium, and thus, $p = 1$. Likewise, free entry and zero-profit in the DM with competitive search will imply that

$$r(x, q) := s(\theta(x, q))[x - c(q)] - k \leq 0, \qquad \text{and,} \qquad \theta(x, q) \geq 0, \tag{2.9}$$

where the weak inequalities would hold with complementary slackness: For a submarket (x, q) such that $r(x, q) < 0$, the firm optimally chooses not to post in the submarket. If $r(x, q) = 0$, then the firm is indifferent to creating different numbers of trading posts in submarket (x, q). We can also deduce that $r(x, q) > 0$ cannot be an equilibrium: If expected profit is positive, then this implies $\theta(x, q) = +\infty$, and thus $s(\theta(x, q)) = 0$ which yields a contradiction to the case.[20] We will restrict attention to an equilibrium where Equation (2.9) also holds for submarkets not visited by any buyer.[21]

From (2.9), we can deduce that

$$s(x, q) \equiv \mu \circ b(x, q) = \begin{cases} \frac{k}{x-c(q)} & \iff x - c(q) > k \\ 1 & \iff x - c(q) \leq k \end{cases}. \tag{2.10}$$

Observe that the firm's probability of matching with a buyer, $s(x, q) := \rho(\theta(x, q))$ depends only on the posted terms of trade (x, q). Likewise, the buyer's probability of matching with a firm, $b(x, q) := \lambda(\theta(x, q))$, given the matching technology $\mu : [0, 1] \to [0, 1]$. Thus, in any submarket with positive measure of buyers, the market tightness, $\theta(x, q) \equiv b(x, q)/s(x, q)$, is neccessarily and sufficiently determined by free entry into the submarket. Moreover, the terms of trade of a submarket (x, q) is sufficient to identify the submarket. This will imply that firms' and agents' optimal decision processes do not depend on the equilibrium distribution of agents. They will only depend on the distribution only through the aggregate statistic ω as a result of inflation. Looking ahead, the equilibrium will be (partially) block recursive.

In equilibrium, there is a relation between q and (x, b). That is, in any equilibrium, each active trading post will produce and trade the quantity:

$$q = Q(x, b) \equiv c^{-1}\left[x - \frac{k}{\mu(b)}\right], \tag{2.11}$$

given payment x and its matching probability $s = \mu(b)$. This relation will allow us to perform a change of variables, and re-write the buyers' problems below in terms choices over (x, b), instead of over (x, q).

[20] If we let $(x, q) \mapsto N(x, q)$ denote the equilibrium distribution of trading post across submarkets, condition (2.9) implies that aggregate profit in the DM is zero: $\int \{s(x, q)[x - c(q)] - k\} \, dN(x, q) = 0$.

[21] Justification for this off-equilibrium-path restriction can be rationalized via a "trembling-hand" sort of argument: Suppose there is some exogenous perturbation that induces an infinitesimally small measure of buyers to show up in every submarket. Given a non-zero measure of buyers present in a submarket, if firms' expected profit is still negative in that submarket, i.e., $r(x, q) < 0$, then the market will not be active. This restriction is commonly used in the directed search literature (see, e.g., Menzio et al., 2013; Acemoglu and Shimer, 1999; Moen, 1997).

2.4.2 Equilibrium CM individual

Let us denote $\mathcal{C}[0,\bar{m}]$ as the set of continuous and increasing functions with domain $[0,\bar{m}]$. Then $\mathcal{V}[0,\bar{m}] \subset \mathcal{C}[0,\bar{m}]$ denotes the set of continuous, increasing and concave functions on the domain $[0,\bar{m}]$. We have the following observations of any CM individual's value and policy functions, which apply to both a steady-state equilibrium or along a dynamic equilibrium transition. (Since these are standard dynamic programming result, proofs of these results are relegated to the Online Appendix.)

Theorem 1. *Assume $\tau/\omega < \bar{m}$. For a given sequence of prices $\{\omega, \omega_{+1}, ...\}$, the value function of the individual beginning in the CM, $W(\cdot,\omega)$, has the following properties:*

1. *$W(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$, i.e., it is continuous, increasing and concave on $[0,\bar{m}]$. It is linear on $[0,\bar{m}]$.*
2. *The partial derivative functions $W_1(\cdot,\omega)$ and $\bar{V}_1(\cdot,\omega_{+1})$ exist and satisfy the first-order condition*

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega y^{\star}(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\begin{cases}\leq 1, & y^{\star}(m,\omega)\geq 0\\ \geq 1, & y^{\star}(m,\omega)\leq y_{\max}(\omega;\tau)\end{cases}, \tag{2.12}$$

   and the envelop condition:

$$W_1(m,\omega) = 1, \tag{2.13}$$

   where $y^{\star}(m,\omega) = m + l^{\star}(m,\omega) - C^{\star}(m,\omega)$, $l^{\star}(m,\omega)$ and $C^{*}(m,\omega)$, respectively, are the associated optimal choices on labor effort and consumption in the CM.

3. *The stationary Markovian policy rules $y^{\star}(\cdot,\omega)$ and $l^{\star}(\cdot,\omega)$ are scalar-valued and continuous functions on $[0,\bar{m}]$.*
   (a) *The function $y^{\star}(\cdot,\omega)$, is constant valued on $[0,\bar{m}]$.*
   (b) *The optimizer $l^{\star}(\cdot,\omega)$ is an affine and decreasing function on $[0,\bar{m}]$.*
   (c) *For every (m,ω), the optimal choice $l^{\star}(m,\omega)$ is finite valued: $0 < l_{\min} \leq l^{\star}(m) \leq l_{\max}(\omega;\tau) < +\infty$, where there is a very small $l_{min} > 0$ and $l_{\max}(\omega;\tau) := y_{\max}(\omega;\tau) + U^{-1}(1) < 2U^{-1}(1) \in (0,\infty)$.*

In the proof to Theorem 1, we also derive the equilibrium decisions of the CM agent. We show that in an equilibrium, CM consumption is

$$C^{\star}(m,\omega) \equiv \bar{C}^{\star} = (U_1)^{-1}(1), \tag{2.14}$$

a finite and non-negative constant. Equilibrium CM asset decision will depend on the aggregate state ω, i.e.,

$$y^{\star}(m,\omega) = \bar{y}^{\star}(\omega) \tag{2.15}$$

and this satisfies the first-order condition (2.12). Finally, from the budget constraint, we can obtain the equilibrium labor supply function as

$$l^{\star}(m,\omega) = \bar{C}^{\star} + \bar{y}^{\star}(\omega) - m. \tag{2.16}$$

Note that $l^{\star}(m,\omega)$ is single-valued, continuous, affine and decreasing in m.

2.4.3 Equilibrium DM buyer

Observe that since $\bar{V}(\cdot,\omega), W(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$ (i.e., are continuous, increasing and concave), then by (A.1), $\bar{V}(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$. In an equilibrium, the DM buyer's problem in (2.5) can be re-written as

$$\begin{aligned} B(\mathbf{s}) = \max_{x\in[0,m],b\in[0,1]} \Bigg\{ & \beta(1-b)\left[\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\omega)},\omega_{+1}\right)\right] \\ & + b\left[u\left(Q(x,b)\right) + \beta\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]\Bigg\}. \end{aligned} \tag{2.17}$$

It appears as if the buyer is choosing his matching probability b along with payment x. However this is just a change of variables utilizing the equilibrium relation (2.11) between quantity q and terms of trade (x,b). From this we can begin to see that there will be a trade-off to the buyer, in terms of an extensive margin (i.e., trading opportunity b), and, an intensive margin (i.e., how much to pay x).

The operator defined by (2.17) clearly does not preserve concavity: The objective function in (2.17) is not jointly concave in the decisions (x,b) and state m, since it is bilinear in the function b and the value function $\bar{V}$, or the flow payoff function u. However, we can still show that the resulting DM buyers' optimal choice functions for (x,b), denoted by $(x^{\star},b^{\star})$, are monotone, continuous, and have unique selections, using lattice programming arguments.

The following theorem summarizes the properties of a DM agent's value and policy functions:[22]

Theorem 2 (DM value and policy functions)**.** *For a given sequence of prices $\{\omega,\omega_{+1},...\}$, the following properties hold.*

[22]Theorem 2 is a generalization of the observation of Menzio et al. (2013) in two aspects: (i) We have additional endogenous CM participation in our model; and (ii) the theorem extends beyond steady state equilibria to encompass equilibrium along a dynamic transition.

1. *For any $\bar{V}(\cdot, \omega_{+1}) \in \mathcal{V}[0, \bar{m}]$, the DM buyer's value function is increasing and continuous in money balances, $B(\cdot, \omega) \in \mathcal{C}[0, \bar{m}]$.*

2. *For any $m \leq k$, DM buyers' optimal decisions are $b^{\star}(m, \omega) = x^{\star}(m, \omega) = q^{\star}(m, \omega) = 0$, and $B(m, \omega) = \beta\bar{V}[\phi(m, \omega), \omega_{+1}]$, where $\phi(m, \omega) := (\omega m + \tau) / [\omega_{+1}(1 + \tau)]$.*

3. *At any (m, ω), where $m \in [k, \bar{m}]$ and the buyer matching probability is positive $b^{\star}(m, \omega) > 0$:*

   (a) *The optimal selections $(x^{\star}, b^{\star}, q^{\star})(m, \omega)$ and $\phi^{\star}(m, \omega) := \phi[m - x^{\star}(m, \omega), \omega]$, are unique, continuous, and increasing in m.*

   (b) *The buyer's marginal valuation of money $B_1(m, \omega)$ exists if and only if $\bar{V}_1[\phi(m, \omega), \omega]$ exists.*

   (c) *$B(m, \omega)$ is strictly increasing in m.*

   (d) *the optimal policy functions $b^{\star}$and $x^{\star}$, respectively, satisfy the first-order conditions*

$$u \circ Q[x^{\star}(m, \omega), b^{\star}(m, \omega)] + b^{\star}(m, \omega)(u \circ Q)_2[x^{\star}(m, \omega), b^{\star}(m, \omega)] \\ = \beta[\bar{V}(\phi(m, \omega), \omega_{+1}) - \bar{V}(\phi^{\star}(m, \omega), \omega_{+1})], \quad (2.18)$$

and,

$$(u \circ Q)_1[x^{\star}(m, \omega), b^{\star}(m, \omega)] = \frac{\beta}{1 + \tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1[\phi^{\star}(m, \omega), \omega_{+1}]. \quad (2.19)$$

We prove these results in sequence, in Online Appendix D. Here, we summarize briefly the workings behind these results: Part 1 of the Theorem uses standard results from optimization and can be found in Lemma 1 of the appendix. Part 2 is proven as Lemma 2 in the appendix, and its insight here is simple: If buyers do not carry enough money to at least pay for a trading post's fixed cost, no firm will want to set up that post in equilibrium, and so the buyers get nothing. Part 3(a) is proven as Lemma 3 using the fact that a log-transform of the DM buyer's objective function is jointly concave in the choice variables (x, b), and is continuous in m (fixing the aggregate state). It nevertheless satisfies an increasing difference—and therefore, supermodularity—property. Thus, by lattice programming arguments, we can show that the DM buyer's optimal policies are increasing in m. Lemmata 4 and 5 in the appendix, together establish Parts 3(b) and 3(c): These show that whenever a buyer has a chance of matching, her value function is differentiable. As a result, we can also characterize her best response in terms of a matching probability (extensive margin) and spending level (intensive margin) via Euler equations in Part 3(d), and this is proven in Lemma 6.

2.4.4 Market clearing

Goods in CM. In equilibrium, the total production of CM good equals its demand:

$$Y = C \equiv U^{-1}(1).$$

Goods in DM. Given equilibrium policy functions, $x^{\star}$ and $b^{\star}$, and, equilibrium distribution of money G and wage ω, Equation (2.11) pins downs market clearing for each submarket in the set of equilibrium submarkets $\{(x(m,\omega), b(m,\omega)) : m \in \text{supp}(G(\cdot,\omega))\}$.

Money demanded must also equal money supplied:

$$\frac{1}{\omega} = \int m \mathrm{d}G(m;\omega) > 0. \tag{2.20}$$

Since M is the beginning of period aggregate stock of money in circulation, then the LHS of (2.20), $1/\omega = M \times 1/\omega M$, is the beginning of period real aggregate stock of money, measured in units of labor. The RHS of (2.20) is beginning of period aggregate demand, or holdings, of real money balances measured in the same unit.

2.5 Existence of a SME with unique distribution

For the rest of the paper, we focus on a stationary monetary equilibrium (SME), which comprises the characterizations from Section 2.4, where the sequence of prices are constant: $\omega = \omega_{+1}$.

Definition 1. A *stationary monetary equilibrium* (SME), given exogenous monetary policy τ, is a

- list of value functions $\mathbf{s} \mapsto (W, B, \bar{V})(\mathbf{s})$, satisfying the Bellman functionals: (2.4), (2.5), and jointly, (A.1)-(2.6);
- a list of corresponding decision rules $\mathbf{s} \mapsto (l^{\star}, y^{\star}, b^{\star}, x^{\star}, q^{\star}, z^{\star}, \pi^{\star})(\mathbf{s})$ supporting the value functions;
- a market tightness function $\mathbf{s} \mapsto \mu \circ b^{\star}(\mathbf{s})$ given a matching technology μ, satisfying firms' profit maximizing strategy (2.10) and (2.11) at all active trading posts;
- an ergodic distribution of real money balances $G(\mathbf{s})$ satisfying an equilibrium law of motion

  $$G(E) = T(G)(E) := \int P(\mathbf{s}, E)\mathrm{d}G(\mathbf{s}), \qquad \forall E \in \mathcal{B}(S), \tag{2.21}$$

  where $\mathcal{B}(S)$ is the Borel σ-algebra generated by open subsets of the product state space S, and, $\mathbf{s} \mapsto P(\mathbf{s}, \cdot)$ is a Markov kernel induced by $(l^{\star}, x^{\star}, q^{\star}, z^{\star}, \pi^{\star})$ and $\mu \circ b^{\star}$under τ; and,

- a wage rate function $\mathbf{s} \mapsto \omega(\mathbf{s})$ satisfying the money stock adding up condition (2.20).

At this point, we note that it will not be difficult to show that there is a unique distribution of agents in a SME. However, whether a SME is unique remains elusive to us as the frequency function $\mathrm{d}G(m;\omega)$ does not admit a closed form expression in terms of known functions, and in general, it will also depend on the equilibrium candidate ω. This statement is also true for the original Menzio et al. (2013) setting, if the authors' model had money supply growth. The intractability of their version of the frequency function $\mathrm{d}G(m;\omega)$ under money supply growth comes about from the modeller no longer being able to work out analytically how long it will take for DM-unmatched buyers' balances to get eroded by inflation, before they have to go to work again. In contrast, the variation in Sun and Zhou (2018) admits an analytical form for $\mathrm{d}G(m;\omega)$ and as a result they can show that there is a unique SME. This special result arises from their assumption that all types of agents in the DM must deterministically enter the CM *after one round* of trade (or no trade) in the DM, so that the aggregate demand for money in their model can be analytically described by a composition of equilibrium decision functions with well-behaved properties and an assumed exogenous distribution of CM preference shocks. In their model, without an exogenous distribution of CM preference shocks, given the market timing assumptions, there would be no distribution of agents since preferences are quasilinear in their CM.

Our setting yields a modeling trade-off in the opposite direction: In contrast to Sun and Zhou (2018), we do not require the latter assumption to preserve distributional nondegeneracy. However, our relaxation here would come at an analytical cost on the form of the frequency function $\mathrm{d}G(m;\omega)$. In our opinion, the loss of tractability in this respect is not too severe: Our equilibrium characterization remains computational feasible. In fact, it retains the feature that agents' decision rules depend on the aggregate state only insofar as the scalar aggregate variable, ω. Unlike heterogeneous-agent random matching models, the market clearing conditions in competitive search do not require the conjecture of an entire distribution of assets in order to pin down terms of trade. In that sense, our algorithm for finding a SME will be similar to that used for computing neoclassical heterogeneous-agent models. In fact, with aggregate shocks (e.g., to τ) our setting will also imply an accurate application of the (originally heuristic) Krusell and Smith (1998) algorithm to an exact problem. (A similar point was previously discussed in Menzio et al. 2013, pp.2294-2295.)

The following theorem ensures that in our computations below there exists a steady state, stationary monetary equilibrium, and for each steady state equilibrium ω, there is a unique distribution of agents.

Theorem 3. *There is a SME with a unique nondegenerate distribution G.*

We prove this in Online Appendix E. The idea here is to first show that a composite Bellman functional for each agent satisfies Banach's fixed point theorem. Then, from Theorems 1 and 3, we know that agents' decision functions are monotone and continuous. This implies that for fixed ω, the equilibrium Markov operator on a current distribution of agents G is a monotone map and satisfies measurability conditions. We can easily argue that there monotone mixing as a result of the equilibrium self-map (2.21) on the space of distributions G, and conclude that there is a unique fixed point (in a weak-convergence sense). Finally we show that there is at least one fixed point in the space of ω satisfying the SME conditions by utilizing the intermediate value theorem.

3 Quantitative analyses

Finding a SME requires numerical computation. In this section, we calibrate the model to the US economy and explain a little the equilibrium properties of the benchmark calibrated model. We will pause to discuss the underlying forces and trade-offs at work that will help us to understand the SME outcomes. This will also help guide our understanding at the end of this section, where we provide some comparative SME analyses in terms of allocative, distributional and welfare outcomes.

3.1 Statistical calibration

The CM and DM preference functions are, respectively,

$$U(C) - h(l) = \frac{C^{1-\sigma_{CM}}}{1-\sigma_{CM}} - l, \text{ and, } u(q) = \bar{U}_{DM}\left[\ln\left(q + \underline{q}\right)\text{-}\ln\left(\underline{q}\right)\right],$$

where $\underline{q} = 1 \times 10^{-8}$. Following Menzio et al. (2013), the matching function is such that a trading post's matching probability as a function of a buyer's matching probability is $\mu(b) = 1 - b$.[23]

All the parameters of the model are listed in Table 1. The parameters τ and β are externally pinned down. The rest $(\chi, \sigma_{CM}, k, \bar{U}_{DM})$ are jointly calibrated to match an empirical money demand curve, price dispersion and labor hours statistics. Next, we explain intuitively our identification and calibration strategy.

[23] This function is derived from the well-known telegraph- or telephone-line matching function (see, e.g., Stevens, 2007; Cox and Miller, 1965) that is used in models of labor matching (see, e.g., Petrosky-Nadeau and Zhang, 2017; Hagedorn and Manovskii, 2008; den Haan et al., 2000). Let $\mathscr{B}$ denote the number of buyers, $\mathscr{S}$ the number of sellers at a trading post and v the number of matches that has occurred. A buyer's matching probability is thus $b = v/\mathscr{B}$. A seller's is $s = v/\mathscr{S}$. This implies that $\mathscr{B}/\mathscr{S} = s/b$. The number of matches in a telephone-line matching function is given by $v = \mathscr{B}\mathscr{S}/\left(\mathscr{B}^{\rho} + \mathscr{S}^{\rho}\right)^{1/\rho}$, where $\rho > 0$. From these, we can derive the relationship: $s \equiv \mu(b) = (1 - b^{\rho})^{1/\rho}$. In robustness checks, our results do not change qualitatively with ρ. Hence we chose not to calibrate ρ and instead normalized it as $\rho = 1$.

Table 1: Benchmark estimates

Parameter	Value	Empirical Targets	Description
$1+\tau$	$(1+0.0089)^{1/4}$	Inflation rate[a]	Inflation rate
$1+i$	$(1+0.0385)^{1/4}$	3-month T-bill rate[a]	Nominal interest rate
β	0.99879	-	Discount factor, $\frac{(1+i)}{(1+\tau)}$
χ	0.00297	Aux reg. $(i, M/PY)^b$	CM participation cost
σ_{CM}	2	Aux reg. $(i, M/PY)^b$	CM risk aversion
k	0.003997	Price dispersion (21% s.d.)[c]	Price-posting cost
$\bar{U}_{DM}$	407.77	Mean Hours ($\frac{1}{3}$)	Preference scale

[a] Mean nominal interest and inflation rates in the data are annual.
[b] The auxiliary statistics (data) are from a spline function fitted to the data on annual observations of the (3-month T-bill) nominal interest rate (i) and Lucas-Nicolini New-M1-to-GDP ratio (M/PY).
[c] Kaplan-Menzio dataset: generic-brand aggregation pricing dispersion (21 percent standard deviation).

External calibrations. The benchmark SME inflation rate τ is estimated by the sample mean of long-run (1915-2007) CPI inflation data obtained from FRED (CPIAUCNS). Given the sample mean of the three-month Treasury Bill rate (i) (sourced from the dataset of Lucas and Nicolini, 2015), we can pin down an estimate of the discount factor β using Fisher's *ex-post* relation: $\beta = (1+i) / (1+\tau)$.

Money demand: Identification and internal calibrations. In the model, we can identify the taste parameter σ_{CM} and the cost parameter χ from the observed aggregate money demand relationship. The risk aversion parameter σ_{CM} affects money demand through the individual money demand condition in Equation (2.12), its related envelop condition embedded in marginal continuation value function $\bar{V}_1$ and through aggregation in the overall SME. From Theorem 1, $\bar{V}_1$ depends on probable ex-post DM or CM outcomes. Hence, *ex-ante* $\bar{V}_1$ depends on DM and CM preferences. That is, the CRRA parameter σ_{CM} influences equilibrium money demand.[24]

Likewise, from Equation (2.6) the *ex-post* market participation problem depends on cost parameter χ. In turn, this influences ex-post participation value function $\tilde{V}$. This feeds through to $\bar{V}_1$ through the ex-ante lottery problem in Equation (2.8) and the optimal money demand condition in Equation (2.12).

Since we are focusing on a notion of long-run equilibrium, we calibrate the pair (σ_{CM}, χ) to minimize the distance between the model-implied aggregate money demand relationship and an auxiliary (spline) money-demand model. The auxiliary model is fitted to

[24] Readers familiar with (Lagos and Wright, 2005)-type models might expect calibrations in terms of a corresponding CRRA parameter σ_{DM} instead of σ_{CM}. This is because in (Lagos and Wright, 2005)-type models, agents typically move in and out of the DM together and deterministically. In our setting, there is an individual choice on stochastic transitions between the two markets. It is because of this that there is a direct link from σ_{CM} to the characterization of money demand in Equation (2.12). We chose to normalize $\sigma_{DM} = 1$. In short, it does not matter if we had freed up the DM utility-of-consumption CRRA σ_{DM} for calibration in lieu of σ_{CM}.

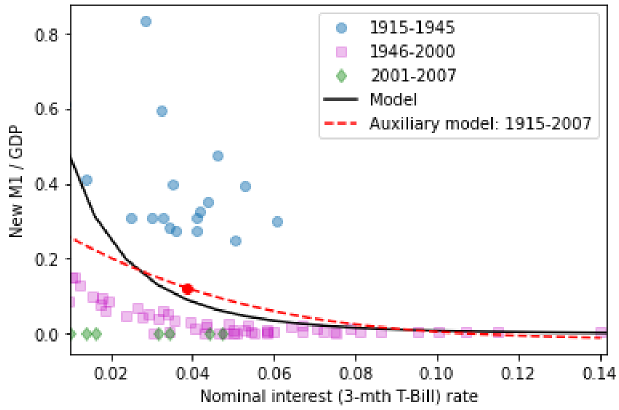

Figure 5: Lucas and Nicolini (2015) money demand annual data (1915-2007), model (green-dashed line) and auxiliary regression model target (red-dashed line). The red dot refers to the sample average for nominal interest.

long-run data spanning from 1915 to 2007.[25] Our focus on using long-run data is similar in spirit to Lucas (2000).

Figure 5 shows the model's aggregate money demand curve (solid black line), with a three-month Treasury-bill measure of the nominal interest rate (i) and the Lucas and Nicolini (2015) "New" M1-to-GDP ratio ($M1/PY$), respective on the horizontal and vertical axes. The long-run data is shown as scatter points with various shapes: circles for pre-WWII observations, squares for post-WWII and pre-Great-Recesssion observations. The dashed line is the auxiliary, empirical money demand curve used as our target for indirectly estimating the model's money demand (solid curve). From the scatter plots, we can deduce that the empirical money demand has shifted in several regimes in the historical data (see also, Ireland, 2009). In following Lucas (2000), we can think of our approach, as specifying a model-implied money demand curve that is a "halfway-house" between these different historical episodes. Indeed, from Figure 5, we can see that the solid curve (model) lies in between the various sub-samples and is close to the empirical (auxiliary) money demand curve.

Hours worked and price dispersion: Identification and internal calibrations. We identify the preference scaling parameter $\bar{U}_{DM}$ (i.e., relative size of DM and CM payoffs) from empirically measured hours worked. In the model, $\bar{U}_{DM}$ is related to the marginal utility function U_1 via Equation (2.12) for individual money demand and Equation (2.16) for an individual's CM budget constraint. That is, $\bar{U}_{DM}$ influences individual optimal labor supply. Through SME, $\bar{U}_{DM}$ is identified from average labor hours of 0.33 in the U.S. data.

Given the Menzio et al. (2013) matching technology μ (with no free parameter), the

[25]The 2007 data measurement preceded the start of the Great Recession.

cost of creating a trading post, k can be pinned down in an SME via firms' profit maximizing strategy (2.10) and (2.11) at all active trading posts. All else equal, k affects the equilibrium location or dispersion of submarkets. As such, we can identify k from measured price-dispersion data. Since we do not have matching historical measures of price dispersion, as a next-best option, we calibrate k to target more recent evidence on price dispersion in micro-level data. For the empirical evidence, we consult with a recent study using price-scanner data in the U.S. by Kaplan and Menzio (2015). According to Kaplan and Menzio (2015) their big-data sample of prices exhibit dispersion. Measured in terms of standard deviation, price dispersion ranges from 19 percent (if goods are defined according to their universal product codes) to 36 percent (if goods are aggregated with different name brands and sizes). A generic-brand aggregation would imply a pricing distribution with about 21 percent in terms of standard deviation. The benchmark calibrated model implies a price dispersion (standard deviation) statistic of 21.7 percent.

3.2 Benchmark SME

In Figure 6, we plot the SME value functions (V, B, W) in the benchmark economy. In the benchmark economy, our algorithm finds that two lottery segments exist. The solid blue line is the graph of $W(\cdot, \omega)$. The dashed green line is the graph of $B(\cdot, \omega)$. The upper envelop of these two graphs give us $\tilde{V}(\cdot, \omega)$, the thick solid green line. Denote conv $\{\cdot\}$ as the convex-hull set operator. The solid magenta graph is the graph of $V(\cdot, \omega)$ obtained through our convex-hull approximation scheme, once we have located all the intersecting coordinates between the set graph $[\tilde{V}(\cdot, \omega)]$ and the upper envelope of the set conv $\{\text{graph}[\tilde{V}(\cdot, \omega)], (0, 0), (\bar{m}, 0)\}$.

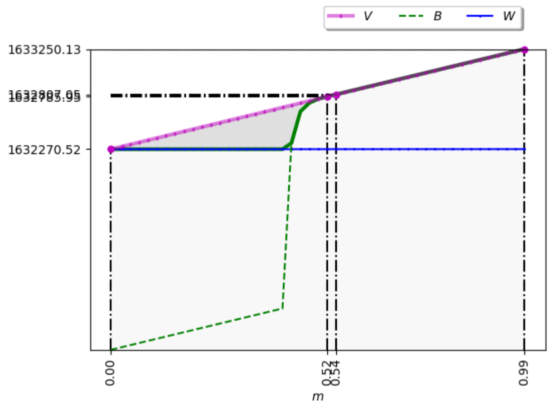

Figure 6: Value functions for benchmark economy.

Sustaining the equilibrium value functions are the policy functions $(l^\star, b^\star, x^\star, q^\star)$, and the lottery policies $(\pi_1, 1 - \pi_1)$ and $(\pi_1', 1 - \pi_1')$ over the prize supports (z_1, z_2) and (z_1', z_2'), where $\pi_1(m, \omega) = (z_2 - m) / (z_2 - z_1)$ and $\pi_1'(m, \omega) = (z_2' - m) / (z_2' - z_1')$.

The other policy functions can be seen in Figure 7. Consider the panel depicting the graph of the CM labor supply function. As shown earlier in (2.16), labor supply is affine and decreasing in money balance. There are three shaded patches in the Figure's panels. The darker (and narrowest) patch corresponds to the region where $m \in [0, k)$, i.e., an agent will never match nor trade in the DM. The orange patches (one of which overlaps the dark-red patch) are the regions of the agent's state space in which a lottery may be played—i.e, $[z_1, z_2]$ and $[z_1', z_2']$. What matters for each agent in the SME is then the loci of these policy functions outside of the orange patch, but including the points on its boundary. These will be consistent with the equilibrium's ergodic state space of agents. As proven in Theorem 2, the policy functions $(b^\star, x^\star, q^\star)$ are monotone in m in the relevant subspace where an agent can exist at any point in time. The relevant ergodic subspace of $[0, \bar{m}]$ in equilibrium is given by $\{z_1, [z_2, z_1'], [z_2', \bar{m}]\} = \{0, [0.52, 0.54], [0.98.., \bar{m}]\}$ in the benchmark economy in Figure 6 or Figure 7.

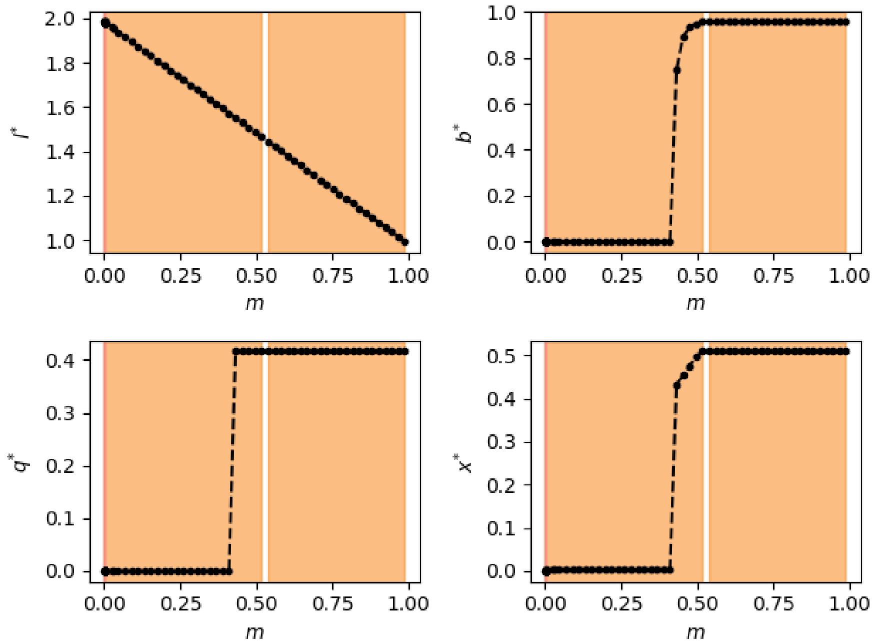

Figure 7: Markov policy functions in the benchmark economy.

Given the information about our benchmark SME's active or relevant agent state space, and, the corresponding policy functions, we can simulate an agent's outcomes and also compute the equilibrium distribution of real money holdings.[26] To do so, one may begin from any initial agent named (m, ω) and apply the decision rules computed earlier, as in Figure 7. Details of the algorithms for simulating the SME outcomes can be

[26]With competitive search, the domain of real money balances will be finite. Menzio et al. (2013) derive a unique closed-form for the graph of the distribution of money holdings, in the special case of zero inflation. When there is non-zero inflation, this becomes analytically intractable. We can numerically compute this, given agents' equilibrium policy functions.

found in our Online Appendix G. Readers interested in the SME behavior of agents may take a detour here to study the benchmark-calibrated SME simulation outcomes in Online Appendix H. Otherwise, we may proceed to discuss the equilibrium trade-offs faced by agents (i.e., the model mechanism) in the next section.

4 Trade-offs: inspecting the mechanism

In this section, we explain the mechanism behind equilibrium behavior and the attendant welfare and redistributive consequences of inflation. We identify the opposing forces underlying an SME as a function of inflation policy τ. The key insight here is that there is an endogenous, *trading-probability* or *extensive margin* that acts in opposite direction to an *intensive-margin* effect of inflation. The latter is a feature that also exists in all other heterogeneous-agent monetary models (including random-matching models). In contrast to standard models or random matching models, competitive search equilibrium induces matching probabilities that are endogenous to individual states and to aggregate (policy) outcomes. This creates a new channel from policy τ to the cross section of money holdings that will work against a standard inequality-reducing effect of τ. For the sake of exposition, we will split our discussion of these two opposing forces according to activities in the CM and the DM.

CM-participation intensive vs. extensive margins. Positive inflation ($\tau > 0$) induces the following trade-offs: On one hand, with inflation, individuals would like to visit the CM more frequently to work and consume there (since in the CM money is not needed for exchange). On the other hand, given a real fixed cost $\chi \geq 0$ of entering the CM, higher inflation means that low-balance agents in the DM will face a greater barrier to engage in liquidity management in the CM. This is because of two possibilities: (i) their natural short-sale constraints in (2.6) may be violated if inflation is too high, i.e., $m - \chi < -y_{max}(\omega; \tau) < 0$, and so they choose to stay in the DM and are more likely to keep realizing a bad draw of the zero balance lottery prize; or (ii) their short-sale constraints in (2.6) are not binding, but the value of going to CM, $W(m - \chi, \omega)$ is still dominated by the value of going to the DM, $B(m, \omega)$. However, in equilibrium, in order to continue deriving consumption value in the DM, an agent would also need to ensure that he has sufficient balance to pay to go back to the CM often enough to maintain enough precautionary saving of money.

These trade-offs imply two margins for a precautionary motive for agents with respect to incomplete consumption insurance: Either they work harder each time in the CM and bear the cost of holding excess money balances (*intensive labor-CM margin*), or, they work less in each CM instance, reduce their spending in each DM exchange, and ensure that they are more likely to be able to afford to go to the CM frequently (*extensive labor-CM margin*).

Remark 2. Recall from Remark 1 that the fixed cost parameter χ is not crucial to the existence of this intensive-versus-extensive margin mechanism in the model. We again emphasize that the device $\chi \geq 0$ is merely to facilitate quantitative fit of the model. We confirm this in a robustness check for the case that $\chi = 0$, in our Online Appendix I.2. There we show that the same qualitative conclusions arise, as in our benchmark-calibrated model results to be discussed next.

DM-specific intensive vs. extensive margins. This is the key trade-off with respect to the DM. This trade-off is not present in standard general-equilibrium or in random-matching models. Consider the equilibrium description of firms' optimal behavior in relation to DM production and profit maximization (2.11). Given the firms' best response in a SME, we can deduce the following about a potential DM buyer: $Q_1(x, b) > 0$, $Q_2(x, b) < 0$, $Q(x, b)$ is weakly concave, and $Q_{12}(x, b) = 0$. In words, we have another tension here: On one hand, faced with a given probability b of matching with a trading post, the more a buyer is willing to pay, x, the more q she can consume. (This is the *intensive margin* of DM trade—i.e., how much one can purchase.) On the other hand, given a required payment, x, a buyer who seeks to match with higher probability, b, must tolerate eating less q (This is the *extensive margin* of DM trade—i.e., trading opportunities.)

From Theorem 2, we know that if a DM buyer brings in more (less) money balance every period, then x will be higher (lower) and b will be higher (lower). The tension just outlined above gives an ambiguous resolution on x or q. Thus the intensive margin faces a countervailing force in the extensive margin within the DM, as this will interact with the CM-participation intensive and extensive margin trade-off as well.

How might inflation affect money holdings inequality? As in standard heterogeneous-agent models, inflation has redistributive-tax effect (through the intensive margin of trade in all markets). Intuitively, agents who have higher (lower) balances have a lower (higher) marginal, ex-ante valuation of money. Inflation thus taxes the "rich" and gives to the "poor". This is also the case in standard heterogeneous-agent monetary models.

However, inflation also raises individuals' downside risks or probabilities of not getting to trade in decentralized search markets. Consider Equations (2.18), (2.19) and Theorem 2 (part 2). With higher inflation, agents tend to carry less money holdings. However, since an individual's probability of not trading in the DM is decreasing in her money balance, this risk affects individual incentives on the extensive margin—i.e., how fast agents trade relative to how much liquidity they carry into in these markets. If individual matching probabilities decline and their dispersion widens as inflation gets higher, then there could be rising inequality in money holdings. From Figure 7, we can see that the equilibrium matching probabilities are concave in money holdings—i.e., with higher inflation, they tend to decline more slowly for the "rich" than the "poor". This works in an opposite direction to the intensive-margin, redistributive effect of inflation. In equilibrium, agents trade off between these two channels. This trade-off depends on long-run inflation targeting policy.

5 Inflation and the opposing intensive-extensive margins

We now use the calibrated model to demonstrate where the intensive-versus-extensive-margin tension resolves, in the face of higher inflation. On the horizontal axes of the figures that follow, we are increasing the (quarterly) steady-state inflation rate, τ, within the set $(\beta - 1, 0.025]$. On the vertical axes, we measure relevant statistics for each corresponding economy under policy τ. In the following discussion, we refer to each equilibrium as SME(τ).

We note that the results below remain qualitatively similar even when we shut down the additional friction of CM participation ($\chi = 0$). This is demonstrated in our Online Appendix I.2. In other words, the extensive margin channel from competitive search matters in terms of creating a countervailing effect to the redistributive effect of inflation. This additional trade-off is nonexistent in standard or random-matching, heterogeneous-agent monetary models.

Money distribution. First, consider how the distribution of end-of-period money holdings in a SME(τ) varies with successively higher trend inflation τ. We see from Figure 8 (*left panel*) that for higher inflation economies, average money balance is smaller. We have checked that this is also true for the entire distribution: Each heterogeneous agent would also be holding less money balance in an SME that has a higher inflation rate. That is, as inflation rises, the cost of holding money increases. In response, agents carry less money out of the CM.

If we consider a "90/10" measure of inequality in Figure 8 (*right panel*)—i.e., the ratio of balances held by the top ten percent to the bottom ten percent of agents in the distribution—then we see that for low levels of How might inflation affect money holdings inequality? inflation, this measure of inequality falls as inflation rises. That is, the decline of money holdings for the top ten percent of money holders (the "rich") is relatively faster than that for the bottom ten percent (the "poor") as inflation rises. This echoes our discussion of the model mechanism earlier. Here, inflation works through a stronger intensive margin. Through the intensive margin, inflation taxes the rich (those with low marginal valuations of money) and redistributes to the poor (those with higher marginal valuations of money). However, there is still the countervailing force arising from the interaction of endogenous trading probabilities with how quickly agents should spend searching for goods in the DM. At some point, as inflation gets high enough, the latter extensive-margin channel begins to dominate the intensive margin. As a result, we see money-holdings inequality rising with inflation.

We see a similar result if we use the Gini measure for money holdings inequality (see Figure 8, *bottom panel*). The green square marker in Figure 8 (*bottom panel*) denotes a reference SME at zero inflation, or at $\tau = 0$. The red diamond marker is at an SME with annual inflation of 10%.

Next, we can further dissect the reason for this non-monotone, "U"-shaped money balance inequality relationship with inflation.

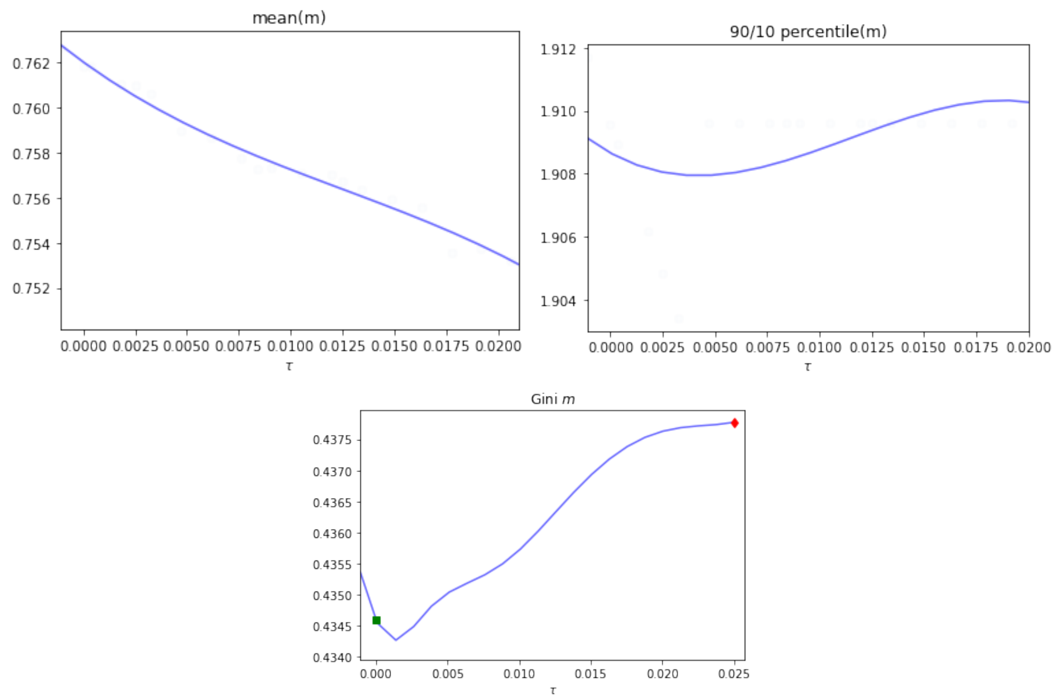

Figure 8: Inflation and money distributions' statistics (*Left*: mean. *Right*: top-10% to bottom-10%. *Bottom*: Gini cofficient.)

Extensive and intensive margins. From Theorem 2, we know that agents' optimal decision rules $(x^{\star}, q^{\star}, b^{\star})$ are monotone increasing in individual real balances. Thus, with higher inflation, money holdings across the distribution tends to fall and agents who match with firms will also be paying $(x^{\star})$ and consuming $(q^{\star})$ less in the DM *on average*. The solid lines in Figure 9 (*top left and right panel*s) illustrate this. The corresponding dashed and circled lines denote the bottom- and the top-ten percent of outcomes, of the respective SME(τ) distributions, at each inflation rate. Consistent with the 90/10 figure for money distribution in Figure 8 (*right panel*), we see that although the upper 10 percent of money holders all decrease money balances, they do so at slower rate than the bottom 10 percent of agents. As a result, the "rich" face a slower decline in their total payments ($x^\star$) for DM goods relative to the "poor" in Figure 9 (*top-left panel*). Also, while matching probabilities of all buyer types fall with inflation, the "rich" experience a slower decline in these probabilities relative to the "poor"—see Figure 9 (*bottom panel*) and Figure 10 (*left panel*)—i.e., there is an increased dispersion in total payments and trading probabilities.

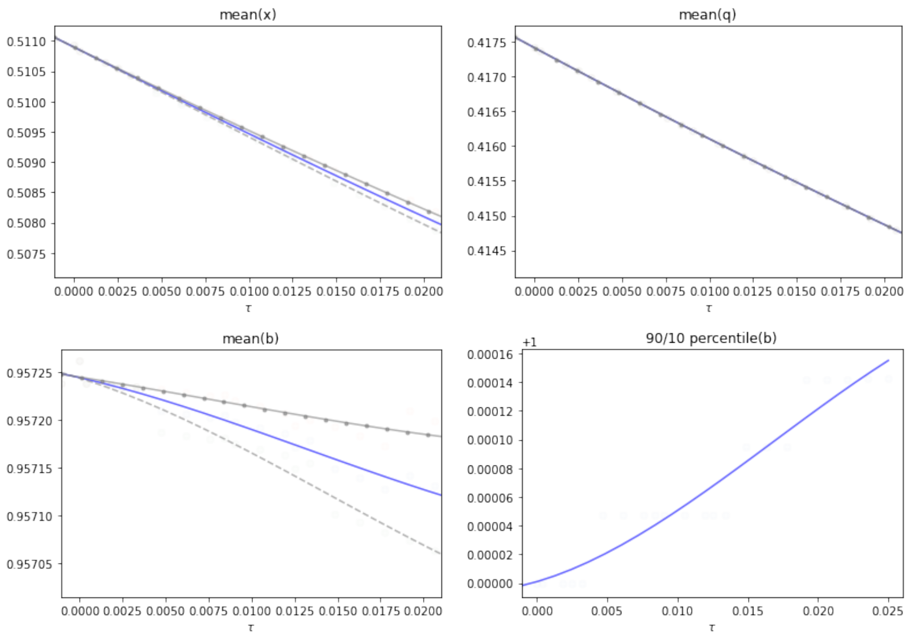

Figure 9: Buyer matching probabilities and quantities—Mean (left panels, solid), 90% (solid-dotted) and 10% (dashed) percentiles.

Reconciling Fact 1 (Inflation and consumption inequality). For low inflation ranges, as inflation rises, the 90/10 inequality measure for DM consumption is rising. For high enough inflation, this inequality measure is falling. This is true if we consider either a 90/10, a standard deviation, or a Gini measure of consumption inequality. Examples of these, respectively, are shown in Figure 10 for either DM consumption inequality or all consumption (DM and CM) inequality. Interestingly, if we consider these equilibria "backwards"—i.e., read Figure 10 from high to successively lower inflation outcomes—we see that the model predicts a hump-shaped inflation-consumption-inequality relation. This is consistent with observed long-run correlations between inflation and consumption inequality (recall **Fact 1** in Section 1.2). The correlation had been positive until in recent years, when that has become negative, while inflation had been steadily declining.

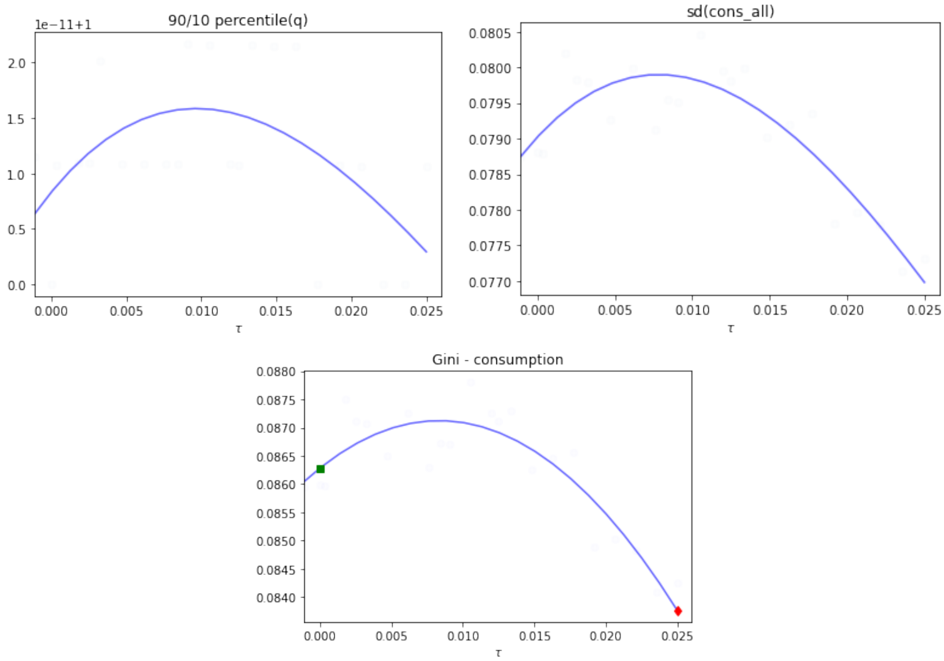

Figure 10: Inflation and consumption inequality measures—90/10 ratio and standard deviation

Reconciling Facts 1 and 2 (Inflation, consumption inequality and price dispersion). As shown in Figure 11, this is driven by rising (implicit) prices and dispersion of prices as inflation goes up. This is because in equilibrium, DM agents with more money end up paying more and consuming more, relative to those who have less money. For sufficiently low inflation, as inflation rises the dispersion in prices increases; and the increase is steeper at the top. That is, those with more money optimally direct themselves to trading posts where they end up forking out higher total payments (x). This is positively related with them facing a higher trading probability, *relative to* those who are poorer in money holdings. The "richer" agents face higher prices ($p \equiv x/q$) relative to the "poorer" ones. (This also rationalizes **Fact 2** in Section 1.2.) The higher total payments are induced by relatively higher prices at those trading posts. Thus, as inflation rises, everyone reduces their consumption, but at some point, the "rich" have a reduction in q that is relatively steeper than that for the "poor". That is, the "rich" in successively higher inflation settings are holding relatively more money than the "poor" for the purposes of being able to direct themselves to trading posts that offer a *relatively* higher probability of matching, compared to those frequented by the "poor".

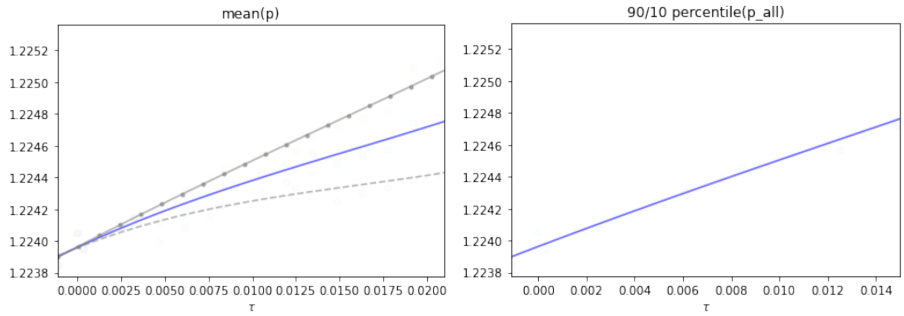

Figure 11: Inflation and price dispersion—Mean (solid), 90% (solid-dotted) and 10% (dashed) percentiles

Reconciling Facts 1, 2 and 3 (Inflation, consumption inequality, price dispersion and speed of monetary transactions). Another way to see the increased dominance of the extensive, trading-opportunity margin is as follows. Consider Figure 12. Since the probability of not getting matched, $1 - b^{\star}(m)$, increases with inflation for all agents in the DM, this exacerbates the cost of holding money for DM buyers who are unmatched, especially those holding higher balances. Even though we observe that the across the distribution of agents, their matching probabilities $b^{\star}$ are decreasing with inflation, what matters for DM agents is how quickly they can dispose of a given amount of liquidity they carry into each DM round, in exchange for DM goods. A useful summary statistic here would be the (average) payment in the DM across buyers, bx/m. We see in Figure 12 (*top-left panel*) that the average measure of total payments for goods, bx, is falling slower than the fall in their average money balances. That is, the average speed of transactions in the DM, the ratio bx/m, is increasing with inflation. This echoes the new DM extensive margin that we identified above. It is also consistent with **Fact 3** in Section 1.2.

At the distribution level though, we again see the intensive-versus-extensive-margins tension through Figure 12 (*top-right panel*). For low inflation, the "rich" tend to transact more slowly relative to the "poor". However, as inflation rises, the extensive margin effect begins to dominate; the "rich", although holding relatively more money, will spend their money holdings on directing themselves to higher matching-probability and higher-priced trading posts. A consequence of this is that agents would also go to the CM more often to manage their liquidity, as shown in Figure 12 (*bottom panel*).

In summary, agents consume less (intensive margins in both markets) in return for being able to trade faster in DM and to visit the CM more often (extensive-margin effects in both markets). In the DM sector, the new extensive margin incorporates the effect that inflation policy has an additional effect on the cross section of money holdings by affecting their heterogeneous matching probabilties. In short, inflation may not necessarily be a redistributive tax that reduces (money) wealth inequality—a feature of earlier heterogeneous-agent monetary models (see, e.g., Molico, 2006; Imrohoroğlu and Prescott,

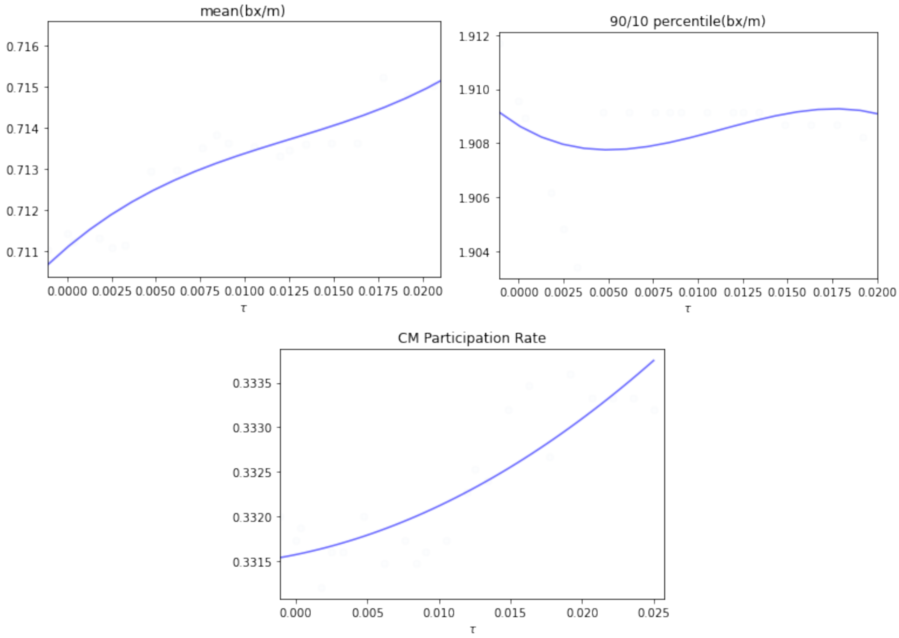

Figure 12: Agents trade faster with money in DM, return to CM quicker

1991b; Erosa and Ventura, 2002). With a trade-off between inflation tax on the intensive margin of allocations and inflation incentivizing agents to trade faster on the extensive margin, we get non-monotone distributional consequences—a U-shaped inflation-money-inequality relationship and a hump-shaped inflation-consumption-inequality relation.

Effect of inflation on welfare. We now turn to the traditional question of how costly is inflation, from the calibrated model's perspective. We measure welfare as how much consumption equivalent variation (CEV) an ex-ante agent is willing to give up in order to move from a zero-inflation economy to a higher-inflation one. This CEV measure falls with inflation.[27]

Figure 13 (*top-left* and *top-right panels*) shows that the welfare cost of inflation rises with inflation, for both average agents and other agents across the respective distributions. Consider the solid line in Figure 13: The *mean* welfare cost of moving the economy from

[27] An individual's *ex-ante* welfare is naturally measured as $Z_\tau := \int \bar{V}(m, \omega(\tau))\, \mathrm{d}G(m, \omega(\tau))$. Consider a reference equilibrium, SME(τ_0). Its corresponding individuals' *ex-ante* value is Z_{τ_0}. In an alternative economy SME(τ), the consumption equivalent variation (CEV) required to move the individual from the reference τ_0-economy to the τ-economy will be defined as

$$CEV(\tau) = \left[\frac{U^{-1}(Z_\tau)}{U^{-1}(Z_{\tau_0})} - 1 \right] \times 100\%.$$

This individual-specific variation is measured in units of the CM good (i.e., labor). In the comparisons, we set $\tau_0 = 0$, i.e., the zero-inflation economy.

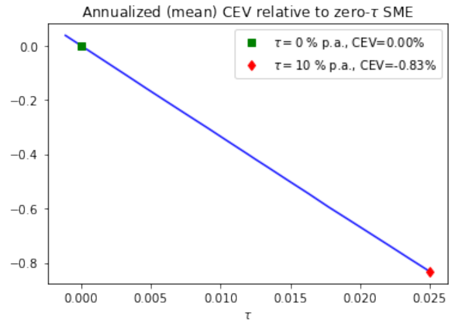

Figure 13: (a) Mean welfare (CEV) falls for all types (0% to 10% inflation *p.a.*).

zero to ten percent inflation *per annum* is about 0.83 percent of consumption loss (relative to the zero-inflation SME mean consumption outcome.

Cross-model welfare cost comparisons. In Table 2, we compare our model's welfare cost of inflation with some well-known studies in the literature. In representative-agent models such as those of Lucas (2000) or Lagos and Wright (2005) (which has additional bargaining frictions), the comparative-steady-state welfare cost of inflation can be quite high. However, this tends to be lower when one revisits the question in a heterogeneous-agent version of the models. It is well known that the redistributive margin of inflation tax is always present in heterogeneous-agent models. This margin tends to reduce the inefficiency of holding money in the presence of inflation (see, e.g., Camera and Chien, 2014; Kocherlakota, 2005; Erosa and Ventura, 2002). This is also the case in random-matching versions of such models (see, e.g., Chiu and Molico, 2010; Molico, 2006). In contrast, in heterogeneous-agent models such as Imrohoroğlu and Prescott (1991b), which has more free parameters to govern frictions, one could obtain a welfare cost of inflation as high as 0.9 percent *per annum*, in CEV terms.

Table 2: Welfare cost (CEV) from 0% to 10% (p.a.) inflation economy.

Economy	**Welfare Cost** (%)[a]	**Remarks**
Benchmark	**0.83 / 1.31**	static / transition
Imhoroğlu-Prescott (1991)	0.90	Bewley-CIA[b]-HA[d]
Chiu-Molico (2010)	0.41	RM[c]-HA[d]
Lagos-Wright (2005)	1.32	RM[c]-RA[e]-TIOLI[f]
Lucas (2000)	0.87	CIA[b]-RA[e]

[a] Annualized CEV cost (relative to zero-inflation economy)
[b] CIA: Cash-in-advance model
[c] RM: Random matching model
[d] HA: Heterogeneous agent model
[e] RA: Representative agent model
[f] TIOLI: Take-it-or-leave-it bargaining

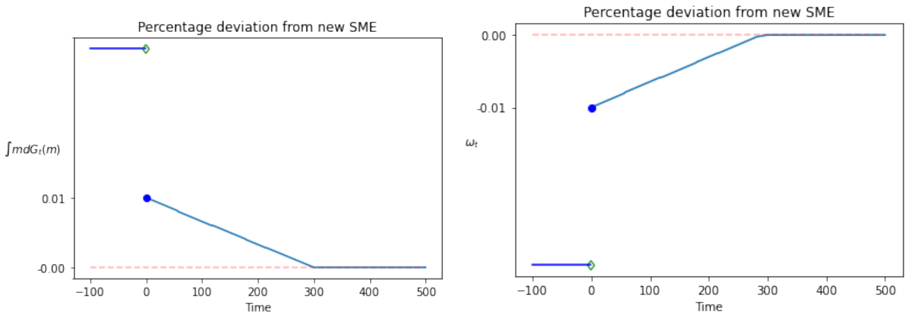

Figure 14: Transition from zero- to ten-percent-inflation SME. *Left*: Aggregate money. *Right*: Real wage rate.

For completeness, we also calculate the (mean) welfare cost of inflation between a zero-inflation and a ten-percent-inflation SME, taking into account the effects of *transitional dynamics*. Figure 14 shows the transition of the aggregate state variable in terms of total money holdings (*left panel*) and its inverse statistic which is the model's real wage rate, ω (*right panel*). The vertical axes are measure in percentage deviations from the respective outcomes in the new or terminal SME. The economy is assumed to be in the initial SME(τ) where money supply growth rate is $\tau_{-1} = 0$ percent. At date $t = -1$, money supply growth rate jumps to $\tau'_{-1} = \tau' = 10$ percent *per annum*. The economy reacts in date $t = 0$ and takes some time to transit to the new SME(τ'). We use a standard shooting algorithm to compute the transition. Total welfare cost of inflation, along the transition is 1.31 percent of the initial SME's consumption, as summarized in Table 2 for our benchmark economy.

5.1 Robustness and variations on the theme

Details for what follows can be found in our Online Appendix I. Specifically, in Online Appendix I.1, we consider two variations or robustness checks on our model assumptions. First, we show that our insights above are robust to alternative parametrization of the fixed-cost parameter χ. (There we show only the case of a doubling of χ, but qualitatively, the baseline results remain across a wide range of χ values. Second, we consider an extreme assumption that agents face a zero-borrowing constraint when overcoming the fixed cost of CM entry, χ: This alternative economy is tantamount to a reparametrization of the borrowing limit (2.7) from the benchmark setting to $y_{\max}(\omega; \tau) = 0$.

Consider the second alternative environment of zero borrowing (when it comes to paying for the CM fixed cost). Given this environment, the results are similar qualitatively across increasing inflation rates. However, when one compares this alternative economy with its benchmark counterpart, at any given level of long-run inflation, we have the following additional insights: In the zero-borrowing-limit economy, average money balance is higher, and, equilibrium extensive margin effects in the DM (i.e., on average how fast agents expend their given DM money holdings) are lower than its corresponding benchmark economy. However, the participation rate in CM is higher, but the Gini index is smaller.

The reason is as follows: In the zero-borrowing economy, agents have a stronger precautionary liquidity-risk insurance motive. Since they cannot borrow to overcome the fixed cost of entering the CM to manage their liquidity needs, then whenever they are in the CM, agents will tend to demand more real balances. Likewise, conditional on being in the DM, agents expect to trade at a lower volume relative to their DM money holdings, as they need to economize on the balance in order to possibly overcome the fixed cost of re-entering the CM. This explains the on-average higher money balance (in comparison to the benchmark economy) and the lower rate of trading in the DM. In return, agents would like to go to the CM more often to demand additional precautionary liquidity. That explains a relatively higher top end of the money distribution relative to the bottom (i.e., a more left-skewed distribution), and hence a lower Gini index, in comparison to the benchmark economy's outcome.

Finally, as alluded to earlier in Section 4, we also show in Online Appendix I.2 that χ *per se* is not needed to materialize the model's endogenous extensive-margin forces. That is, if we set $\chi = 0$, the same qualitative pattern arises as in the benchmark economy discussion in Section 5. It is nevertheless a useful parameter for quantitative reasons.

6 Conclusion

We proposed a theoretical and quantitative heterogeneous-agent monetary model based on Menzio, Shi and Sun (2013) to study the effect of inflation targeting on welfare and inequality. In this paper, we have shown that details matter. They matter, both from a logical-theoretic and from a quantitative perspective, when thinking about market frictions and understanding the effects of monetary policy on heterogeneous individuals. We focused on competitive search and matching in markets when money has value in equilibrium exchange.

We highlight a new mechanism—an endogenous trade-off between intensive and extensive margins—through which monetary policy has impact on the aggregate economy and welfare. In contrast to well-received wisdom that inflation acts as a redistributive tax, we showed that there is a countervailing force with competitive search. Because agent's matching probabilities are endogenous to their states, monetary policy has differential effects on across an equilibrium cross-section of heterogeneous agents. Agents trade-off between their desire to consume more and their desire to be able to trade more frequently, and this notion of frequency now depends on the endogenous matching probabilities.

Quantitatively, the effect of inflation tax on liquid-wealth inequality is non-monotone. Thus, the welfare cost of inflation in our model is still sizable, despite the redistributive effect of inflation that tends to induce heterogenous-agent monetary models to produce lower costs of inflation, relative to their representative-agent counterparts.

In this paper, we deliberately focused on a single-asset, pure-currency economy in order to have a simple and clear understanding of our new equilibrium relation between extensive- and intensive-margins of trade-off, and, the effect of inflation tax on the trade-off. We think that if we allowed agents to hold additional illiquid assets (say, in the centralized markets), this may further exarcerbate the inequality result in our model. We are currently exploring this conjecture in an expanded setting with liquid and illiquid assets, and, further with aggregate dynamics.[28]

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[28]Since the framework renders agents' Markov decision processes independent from the aggregate distribution, but for an aggregate scalar statistic, we will have an exact Krusell and Smith (1998) sort of algorithm for computing equilibria. This is especially pertinent for the extended setting with aggregate dynamics, as the competitive search refinement means that the solution method can be made more efficient and more precise, than models where one has to brute-force approximate distributions when solving agent problems.

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ONLINE APPENDIX

Inflationary Redistribution vs. Trading Opportunities

Omitted Proofs and Other Results

This document is also available from

https://github.com/phantomachine/csim.

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A Extension and special cases

Consider a variation on the benchmark setting in the paper. In particular, suppose that each agent $\mathbf{s} := (m, \omega)$ has an initial value each period as

$$\bar{V}(\mathbf{s}) = \alpha W(\mathbf{s}) + (1-\alpha) V(\mathbf{s}), \tag{A.1}$$

where $V(\mathbf{s})$ is the value of playing a fair lottery $(\pi_1, 1-\pi_1)$ over the prizes $\{z_1, z_2\}$, i.e.,

$$V(\mathbf{s}) = \max_{\pi_1 \in [0,1], z_1, z_2} \left\{ \pi_1 \tilde{V}(z_1, \omega) + (1-\pi_1) \tilde{V}(z_2, \omega) : \pi_1 z_1 + (1-\pi_1) z_2 = m \right\}; \tag{A.2}$$

is a natural upper bound on CM saving (in real money balances).

The difference between (A.1) and (A.2) and their respective counterparts in (2.8) and (2.6) in Section 2.3.3 on page 16 of the paper, is that there is a measure α of agents who will participate in the CM for sure each period. When $\alpha = 0$, we recover the simpler model used in the main paper.

Also, when $\alpha = 0$, there is no fixed cost of entering the CM ($\chi = 0$), $U(C) = 0$ for all C, and the labor utility function $h(l)$ is strictly convex, we recover the original Menzio et al. (2013) model as a special case.

Note that when $\alpha = 1$ (i.e., agents get to enter the CM deterministically), $\chi = 0$ (there is no fixed cost of entering the CM), and the continuation value from CM is a convexification of $B(\cdot, \omega)$, our model becomes a version of Rocheteau and Wright (2005) with competitive search markets.

All proofs (to results in the paper) below are written with the more general case of $\alpha \in [0, 1)$ in mind.

B CM individual's problem

Preliminaries. Consider the feasible choice set for CM saving, y: If $\bar{m}$ is an upper bound on end-of-period balance plus transfer (measured in units of labor), then this gives the bounds on end-of-period money balance plus transfer, in current money value, as:

$$\tau M \leq \omega M y + \tau M \leq \omega M \bar{m},$$

where ωM is current nominal wage. Since there is inflation in nominal wage, then next-period initial balance is current end-of-period nominal money balance normalized by the next period nominal wage $M_{+1}\omega_{+1}$, i.e.,

$$\frac{\tau M}{\omega_{+1}M_{+1}} \leq m_{+1} \equiv \frac{\omega M y + \tau M}{\omega_{+1}M_{+1}} \leq \frac{\omega M \bar{m}}{\omega_{+1}M_{+1}}.$$

Using (2.1), we can re-write the above bounds as

$$0 \leq y \leq y_{\max}(\omega;\tau) := \bar{m} - \frac{\tau}{\omega},$$

which applies in the pair of KKT complementary slackness conditions (2.12).

The upper bound on real, end-of-period money holdings, $\bar{m} \in (0,\infty)$, can be derived as:

$$0 < \bar{m} < (U_1)^{-1}(1) \iff 1 < U_1(\bar{m}) < U_1(0). \tag{B.1}$$

Later, in Part 3 of this section, we show that in any equilibrium $U'(C) = 1$. Assuming $U'(C) = 1 < U_1(\bar{m})$ suffices. Intuitively, this permits an agent to accumulate real balances at the end of each period beyond the level of optimal CM consumption $C^\star = U'^{-1}(1)$. Below, we show that having $\bar{m} \in (0,\infty)$ will ensure that there is always an optimal, largest labor effort that is always finite and that in all dates, money balances are bounded.

The following gives the proof of Theorem 1 on page 19 in the paper.

Theorem 1. Assume $\tau/\omega < \bar{m}$. For a given sequence of prices $\{\omega, \omega_{+1}, ...\}$, the value function of the individual beginning in the CM, $W(\cdot, \omega)$, has the following properties:

1. $W(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$, i.e., it is continuous, increasing and concave on $[0, \bar{m}]$. Moreover, it is linear on $[0, \bar{m}]$.

2. The partial derivative functions $W_1(\cdot, \omega)$ and $\bar{V}_1(\cdot, \omega_{+1})$ exist and satisfy the first-order condition

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega y^{\star}(m,\omega)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\begin{cases}\leq 1, & y^{\star}(m,\omega) \geq 0 \\ \geq 1, & y^{\star}(m,\omega) \leq y_{\max}(\omega;\tau)\end{cases}, \quad \text{(B.2)}$$

   and the envelop condition:

$$W_1(m,\omega) = 1, \quad \text{(B.3)}$$

   where $y^{\star}(m,\omega) = m + l^{\star}(m,\omega) - C^{\star}(m,\omega)$, $l^{\star}(m,\omega)$ and $C^{\star}(m,\omega)$, respectively, are the associated optimal choices on labor effort and consumption in the CM.

3. The stationary Markovian policy rules $y^{\star}(\cdot, \omega)$ and $l^{\star}(\cdot, \omega)$ are scalar-valued and continuous functions on $[0, \bar{m}]$.

   (a) The function $y^{\star}(\cdot, \omega)$, is constant valued on $[0, \bar{m}]$.

   (b) The optimizer $l^{\star}(\cdot, \omega)$ is an affine and decreasing function on $[0, \bar{m}]$.

   (c) Moreover, for every (m, ω), the optimal choice $l^{\star}(m, \omega)$ is interior: $0 < l_{\min} \leq l^{\star}(m) \leq l_{\max}(\omega; \tau) < +\infty$, where there is a very small $l_{\min} > 0$ and $l_{max}(\omega) := \min\left\{\bar{m} - \frac{\tau}{\omega}, \bar{m}\right\} + U^{-1}(1) < 2U^{-1}(1) \in (0, \infty)$.

Proof. (*Part 1*). Consider the individual's problem beginning in the CM (2.4). Since $U_1(C) > 0$ for all C, the budget constraint always binds. Thus we can re-write (2.4) as

$$W(\mathbf{s}) = \max_{(C,y)\in\mathbb{R}_+\times[0,\bar{m}]}\left\{U(C) - [pC + y - m] + \beta\bar{V}\left(\frac{\omega y + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right\}. \quad \text{(B.4)}$$

Let

$$(C^{\star}, y_c^{\star})(m,\omega) \in \underset{(C,y)\in\mathbb{R}_+\times[0,\bar{m}]}{\arg\max}\left\{U(C) - [pC + y - m] + \beta\bar{V}\left(\frac{\omega y + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right\}. \quad \text{(B.5)}$$

From (B.4), it is clear that $W_1(\cdot, \omega)$ exists on $[0, \bar{m}]$, and moreover, we have the envelope condition $W_1(\cdot,\omega) = 1 > 0$. This implies that the value function $W(\cdot,\omega)$ is continuous, increasing and concave in m. Moreover it is affine in m.

(*Part 2*). First, we make the following observations: Since U is strictly concave in C, the objective function is strictly concave in C. Moreover, the objective function on the RHS of (B.4) is continuously differentiable with respect to C. The optimal decision, $C^{\star}(m,\omega)$ satisfies the following Karush-Kuhn-Tucker (KKT) conditions:

$$U_1(C) \begin{cases} = p, & C > 0 \\ < p, & C = 0 \end{cases}. \tag{B.6}$$

In an equilibrium, $p > 0$ will be finite—in fact, $p = 1$. Therefore, $C^{\star}(m,\omega) \equiv \bar{C}^{\star} = (U_1)^{-1}(1)$ is a finite and non-negative constant. Thus, we only have to verify that the optimal decision correspondence, given by $l_c^{\star}(m,\omega) \equiv p\bar{C}^{\star} + y_c^{\star}(m,\omega) - m$ at each (m,ω), exists and is at least a compact-valued and upper-semicontinuous (*usc*) correspondence: Fixing $C = \bar{C}^{\star}$, the objective function on the RHS of (B.4) is continuous and concave on the compact choice set $[0,\bar{m}] \ni y$. By Berge's Maximum Theorem, the maximizer $y_c^{\star}(m,\omega)$, or $l_c^{\star}(m,\omega)$, is compact-valued and *usc* on $[0,\bar{m}]$. (In fact, after we further establish that the derivative $V_1(\cdot,\omega_{+1})$ exists, we show below that it will be constant and single-valued with respect to m.)

Second, we take a detour and show that the derivative $\bar{V}_1(\cdot,\omega_{+1})$ exists, in order to characterize a first-order condition with the respect to y. The results below will rely on the observation that since $V(\cdot,\omega_{+1})$ is a concave, real-valued function on $[0,\bar{m}]$, it has right- and left-hand derivatives (see, e.g., Rockafellar, 1970, Theorem 24.1, pp.227-228). Fix $C^{\star}(m,\omega) \equiv \bar{C}^{\star}$. Since $y_c^{\star}(m,\omega)$ is *usc* on $[0,\bar{m}]$, then for all $\varepsilon \in [0,\delta]$, and taking $\delta \searrow 0$, there exists a selection $y^{\star}(m-\varepsilon,\omega) \in y_c^{\star}(m-\varepsilon,\omega)$ feasible to a CM agent m. Similarly, there is a $y^{\star}(m,\omega) \in y_c^{\star}(m,\omega)$ that is feasible to a CM agent $m-\varepsilon$. Moreover, if $l^{\star}(m,\omega) \in l_c^{\star}(m,\omega)$ is an optimal selection associated with $y^{\star}(m,\omega)$, then for an agent at m,

$$\begin{aligned} W(m,\omega) &= \underbrace{U(\bar{C}^{\star}) - l^{\star}(m,\omega) + \beta\bar{V}\left[\frac{\omega\left[m + l^{\star}(m,\omega) - \bar{C}^{\star}\right] + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}_{\equiv Z[m,y^{\star}(m,\omega)]} \\ &\geq \underbrace{U(\bar{C}^{\star}) - l^{\star}(m-\varepsilon,\omega) + \beta\bar{V}\left[\frac{\omega\left[m + l^{\star}(m-\varepsilon,\omega) - \bar{C}^{\star}\right] + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}_{\equiv Z[m,y^{\star}(m-\varepsilon,\omega)]}; \end{aligned}$$

and, for an agent at $m-\varepsilon$,

$$\begin{aligned}
&W(m-\varepsilon,\omega) \\
&=\underbrace{U(\bar{C}^{\star})-l^{\star}(m-\varepsilon,\omega)+\beta\bar{V}\left[\frac{\omega\left[(m-\varepsilon)+l^{\star}(m-\varepsilon,\omega)-\bar{C}^{\star}\right]+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right]}_{\equiv Z[m-\varepsilon,y^{\star}(m-\varepsilon,\omega)]} \\
&\qquad\geq\underbrace{U(\bar{C}^{\star})-l^{\star}(m,\omega)+\beta\bar{V}\left[\frac{\omega\left[(m-\varepsilon)+l^{\star}(m,\omega)-\bar{C}^{\star}\right]+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right]}_{\equiv Z[m-\varepsilon,y^{\star}(m,\omega)]}.
\end{aligned}$$

Rearranging these inequalities, we have the following fact:

$$\begin{aligned}
&\frac{Z\left[m,y^{\star}(m-\varepsilon,\omega)\right]-Z\left[m-\varepsilon,y^{\star}(m-\varepsilon,\omega)\right]}{m-(m-\varepsilon)} \\
&\qquad\leq\frac{W(m,\omega)-W(m-\varepsilon,\omega)}{m-(m-\varepsilon)}\leq\frac{Z\left[m,y^{\star}(m,\omega)\right]-Z\left[m-\varepsilon,y^{\star}(m,\omega)\right]}{m-(m-\varepsilon)},
\end{aligned}$$

which, after simplifying the denominator and taking limits, yields:

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\left\{\frac{Z\left[m,y^{\star}(m-\varepsilon,\omega)\right]-Z\left[m-\varepsilon,y^{\star}(m-\varepsilon,\omega)\right]}{\varepsilon}\right\} \\
&\leq\lim_{\varepsilon\searrow 0}\left\{\frac{W(m,\omega)-W(m-\varepsilon,\omega)}{\varepsilon}\right\}\leq\lim_{\varepsilon\searrow 0}\left\{\frac{Z\left[m,y^{\star}(m,\omega)\right]-Z\left[m-\varepsilon,y^{\star}(m,\omega)\right]}{\varepsilon}\right\} \\
&\iff \\
&\beta\lim_{\varepsilon\searrow 0}\left\{\frac{\bar{V}\left[\frac{\omega(m+l^{\star}(m-\varepsilon,\omega)-\bar{C}^{\star})+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right]-\bar{V}\left[\frac{\omega(m-\varepsilon+l^{\star}(m-\varepsilon,\omega)-\bar{C}^{\star})+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right]}{\varepsilon}\right\}\leq W_1(m,\omega) \\
&\leq\beta\lim_{\varepsilon\searrow 0}\left\{\frac{\bar{V}\left[\frac{\omega(m+l^{\star}(m,\omega)-\bar{C}^{\star})+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right]-\bar{V}\left[\frac{\omega(m-\varepsilon+l^{\star}(m,\omega)-\bar{C}^{\star})+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right]}{\varepsilon}\right\}.
\end{aligned}$$

Since, from (B.4), $W(\cdot,\omega)$ is clearly differentiable with respect to m, the second term in the inequalities above is equal to the partial derivative $W_1(m,\omega)$, which is constant. As $\varepsilon\searrow 0$, there is a selection $l^{\star}(m-\varepsilon,\omega)\to l^{\star}(m,\omega)$, and, by Rockafellar (1970, Theorem 24.1) the first is the left derivative of $\bar{V}(\cdot,\omega_{+1})$. Moreover, the last term is identical to the first, i.e.,

$$\begin{aligned}
&\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega\left(m^{-}+l^{\star}(m,\omega)-\bar{C}^{\star}\right)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right] \\
&\qquad\leq W_1(m,\omega)\leq\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega\left(m^{-}+l^{\star}(m,\omega)-\bar{C}^{\star}\right)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right].
\end{aligned}$$

Therefore, if the optimal selection is interior, these weak inequalities must hold with equality, so we have the left derivative of $\bar{V}$ with respect to the agent's decision variable y as:

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega y^{\star-}(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right]=W_1(m,\omega).$$

where $y^{\star-}(m,\omega)\equiv m^{-}+l^{\star}(m,\omega)-\bar{C}^{\star}$.

By similar arguments, we can also prove that the right directional derivative of $\bar{V}(\cdot,\omega_{+1})$ exists, and show that the right derivative of $\bar{V}$ with respect to the agents decision y as:

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega y^{\star+}(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right]=W_1(m,\omega),$$

where $y^{\star+}(m,\omega)\equiv m^{+}+l^{\star}(m,\omega)-\bar{C}^{\star}$. From the last two equations, we can conclude that the right and left directional derivatives must agree, and thus, we have the first-order KKT condition (2.12) as, repeated here as

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega y^{\star}(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\begin{cases}\leq 1, & y^{\star}(m,\omega)\geq 0\\ \geq 1, & y^{\star}(m,\omega)\leq y_{\max}(\omega;\tau)\end{cases},$$

where the weak inequalities apply with complementary slackness. Since $\bar{V}$ is strictly concave, the condition above ensures a unique selection $y^{\star}(m,\omega)$ at each state. Also, note that in the previous proof of Part 1), we have established the envelop condition (2.13):

$$W_1(m,\omega)=1.$$

(*Part 3.*) Observe that given the assumption in (B.1), we have (B.6) always binding: $U'(C)=p=1$. Also, observe from(B.6) and (2.12) that an individual's current money holding m and the aggregate state ω have no influence on his optimal decision on consumption, $C^{\star}(m,\omega)=\bar{C}^{\star}$, but that $y^{\star}(m,\omega)=\bar{y}^{\star}(\omega)$. However, from the budget constraint, m clearly does affect the optimal labor decision,

$$\begin{aligned}l^{\star}(m,\omega)&=pC^{\star}(m,\omega)+y^{\star}(m,\omega)-m\\ &\underset{(p=1)}{\equiv}\bar{C}^{\star}+\bar{y}^{\star}(\omega)-m.\end{aligned}\tag{B.7}$$

Clearly, $l^{\star}(m,\omega)$ is single-valued, continuous, affine and decreasing in m.

Finally, we show that the optimal choice of l will always be interior. Evaluating the budget constraint in terms of optimal choices at the current individual state m,

$$l^{\star}(m,\omega)=\bar{y}^{\star}(\omega)-m+\bar{C}^{\star}.$$

Since $m \in [0, \bar{m}]$, then, the minimal l attains when m is maximal at $\bar{m}$, and, $\bar{y}^{\star}(\bar{m}, \omega) = 0$:

$$l_{\min} := \check{l}^{\star}(\bar{m}, \omega) \equiv 0 - \bar{m} + \bar{C}^{\star} > 0.$$

The last inequality obtains from (B.1) which requires $\bar{m} < U^{-1}(1)$, and from optimal CM consumption (2.14) which yields $\bar{C}^{\star} = U^{-1}(1)$ in an equilibrium. The maximal l attains when $m = 0$ and $\bar{y}^{\star}(0, \omega) = y_{\max}(\omega; \tau)$:

$$l_{\max}(\omega, \tau) := y_{\max}(\omega; \tau) - 0 + \bar{C}^{\star} = y_{\max}(\omega; \tau) + U^{-1}(1) < 2U^{-1}(1). \tag{B.8}$$

Clearly, $l_{\max}(\omega, \tau) < +\infty$. If we do not have hyperinflation, or, if transfers are not excessively large—i.e., if $\tau/\omega < \bar{m}$—then, $l_{\max}(\omega, \tau) > 0$ will be well-defined. So if $\tau/\omega < \bar{m}$, then we will have an interior optimizer for all m: $0 < l_{\min} \leq l^{\star}(m) \leq l_{\max}(\omega; \tau) < +\infty$. □

C Limited short-sale constraint and CM participation

Here we derive the short-sale constraint that may bind in the ex-ante market participation problem (2.6) in the paper. Suppose an agent were to participate in the CM with initial asset $a = z - \chi$, where z is his ex-ante money balance, and, χ is the fixed cost (in units of labor) of CM participation. Thus if $a < 0$, the agent is said to be short selling, or issuing an I.O.U.

Recall the CM budget constraint is

$$y + C = l + a.$$

The most negative an asset position the agent can attain in an equilibrium is some $\underline{a}$ such that he must work at the maximal amount $l_{\max}(\omega; \tau)$ and cannot afford to save, $y = 0$. From the budget constraint in such an equilibrium, we have:

$$0 + \bar{C}^{\star} = l_{\max}(\omega; \tau) + \underline{a},$$

which then implies that $\underline{a} = \bar{C}^{\star} - l_{\max}(\omega; \tau)$. From (B.8), we can further obtain the simplified expression $\underline{a} = -y_{\max}(\omega; \tau) \equiv -\min\{\bar{m}, \bar{m} - \tau/\omega\}$. Thus the limited short-sale constraint in (2.6) in the paper.

D DM agent's problem

In this section, we provide the omitted proofs leading up to Theorem 2 on page 20 in the paper. Part 1 of the Theorem is obtained in Lemma 1, Part 2 is proven as Lemma 2. Part

3(a) is proven as Lemma 3. Lemmata 4 and 5 together establish Parts 3(b) and 3(c) of the Theorem. Finally, Lemma 6 establishes Part 3(d) of the Theorem.

D.1 DM buyer optimal policies

Recall the DM buyer's problem from (2.17):

$$B(\mathbf{s}) = \max_{x\in[0,m],b\in[0,1]} \{f(x,b;m,\omega)\},$$

where

$$\begin{aligned} f(x,b;m,\omega) := \beta(1-b)\left[\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \\ + b\left[u^{Q}(x,b)+\beta\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right], \end{aligned}$$

and, we have re-defined the composite function $u\circ Q$ as u^{Q}. Note that we have not explicitly written $f(x,b;m,\omega)$ as depending on ω_{+1} which is taken as parametric. In an equilibrium, ω_{+1} will be recursively dependent on ω, thus our small sleight of hand here in writing $f(x,b;m,\omega)$.

The following Lemmata 1, 2, 3, 4, 5, and 6 make up Theorem 2. Also, these results will rely on the following statements and notations:

1. Assume $\{\omega,\omega_{+1},\omega_{+2}...\}$ is a given sequence of prices.

2. Let

$$\phi(m,\omega) := \frac{\omega m+\tau}{\omega_{+1}(1+\tau)},$$

and,

$$\phi^{\star}(m,\omega) = \phi\left[m-x^{\star}(m,\omega),\omega\right].$$

3. Equivalently define the objective function $f(\cdot,\cdot;m,\omega)$ in the DM buyer's problem (2.17) as follows:

$$\begin{aligned} f(x,b;m,\omega) &= \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) \\ &\quad + b\left[u^{Q}(x,b)+\beta\bar{V}\left(\phi^{\star}(m,\omega),\omega_{+1}\right)-\beta\bar{V}\left(\phi(m,\omega),\omega_{+1}\right)\right]. \\ &\equiv \beta\bar{V}\left(\phi(m,\omega),\omega_{+1}\right)+bR(x,b;m,\omega). \end{aligned} \tag{D.1}$$

Remark. Observe that maximizing the value of the objective function $f(x,b;m,\omega)$ in the

DM buyer's problem (2.17) is equivalent to maximizing the second term, $bR(x, b; m, \omega)$. Note that the function $R(x, b; m, \omega)$ has the interpretation of the DM buyer's surplus from trading with a particular trading post named (x, b), by offering to pay x in exchange for quantity $Q(x, b)$.

Lemma 1. *For any $\bar{V}(\cdot, \omega_{+1}) \in \mathcal{V}[0, \bar{m}]$, the DM buyer's value function is increasing and continuous in money balances, $B(\cdot; \omega) \in \mathcal{C}[0, \bar{m}]$.*

Proof. Since the functions $W(\cdot, \omega_{+1}), V(\cdot, \omega_{+1}) \in \mathcal{C}[0, \bar{m}]$, i.e., are continuous and increasing on $[0, \bar{m}]$, and $\bar{V} := \alpha W + (1 - \alpha V)$, then $\bar{V}(\cdot, \omega_{+1}) \in \mathcal{C}[0, \bar{m}]$. The feasible choice set $\Phi(m) := [0, m] \times [0, 1]$ is compact, and it expands with m at each $m \in [0, \bar{m}]$. By Berge's Maximum Theorem, the maximizing selections $(x^\star, b^\star)(m, \omega) \in \Phi(m)$ exist for every fixed $m \in [0, \bar{m}]$, since the objective function is continuous on a compact choice set (Berge, 1963). Evaluating the Bellman operator (2.17), we have that the value function $B(\cdot, \omega) \in \mathcal{C}[0, \bar{m}]$. $\square$

Lemma 2. *For any $m \leq k$, DM buyers' optimal decisions are such that $b^\star(m, \omega) = 0$ and $B(m, \omega) = \beta \bar{V}[\phi(m, \omega), \omega_{+1}]$, where $\phi(m, \omega) := \frac{\omega m + \tau}{\omega_{+1}(1+\tau)}$.*

Proof. Since a buyer's payment x is always constrained above by her initial money balance m in the DM, it will never be optimal for any firm to trade with such a buyer whose $m \leq k$, as the firm will be making an economic loss. In equilibrium it is thus optimal for a buyer $m \leq k$ to optimally not trade and exit the DM with end-of-period balance as m (i.e., with beginning-of-next-period balance $\phi(m, \omega)$ when inflationary transfers are accounted for). As a result, the continuation value is $\bar{V}[\phi(m, \omega), \omega_{+1}]$, and thus, $B(m, \omega) = \beta \bar{V}[\phi(m, \omega), \omega_{+1}]$, if $m \leq k$. $\square$

Lemma 3. *For any (m, ω), where $m \in [k, \bar{m}]$ and the buyer matching probability is positive $b^\star(m, \omega) > 0$, the optimal selections $(x^\star, b^\star, q^\star)(m, \omega)$ and $\phi^\star(m, \omega) := \phi[m - x^\star(m, \omega), \omega]$ are unique, continuous, and increasing in m.*

Observe that the DM buyer's problem has a general structure similar to that of Menzio et al. (2013). The main difference is in the details underlying the buyer's continuation value function, which in our setting is denoted by $\bar{V}(\cdot, \omega)$. Nevertheless, we still have that $\bar{V}(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$. As a consequence the proof of Lemma 3.3 in Menzio et al. (2013) can be adapted to our setting. For the reader's convenience, we repeat the proof strategy of Menzio et al. (2013) below for our model setting in a few steps:

Proof. The DM buyer's problem (2.17) can be re-written as

$$B(\mathbf{s}) = \beta\bar{V}\left(\phi(m,\omega),\omega_{+1}\right) + \exp\left\{\max_{x\in[0,m],b\in[0,1]}\left\{\ln\left(b\right) + \ln\left[R\left(x,b;m,\omega\right)\right]\right\}\right\}.$$

The optimizers thus must satisfy

$$(x^\star,b^\star)(m,\omega) \in \left\{\arg\max_{x\in[0,m],b\in[0,1]}\left\{\ln\left(b\right) + \ln\left[R\left(x,b;m,\omega,\omega_+\right)\right]\right\}\right\}. \tag{D.2}$$

(*Uniqueness and continuity of policies.*) First we establish that the policy functions are continuous, and, at every state, there is a unique optimal selection: Since $u^Q\left(x,b\right)$ is continuous, jointly and strictly concave in (x,b), and by assumption, $\bar{V}\left(\cdot,\omega\right) \in \mathcal{V}\left[0,\bar{m}\right]$, then

$$R\left(x,b;m,\omega\right) \equiv u^Q(x,b) + \beta\bar{V}\left(\phi^\star(m,\omega),\omega_{+1}\right) - \beta\bar{V}\left(\phi(m,\omega),\omega_{+1}\right)$$

is continuous, jointly and strictly concave in the choice variables (x,b). Also, $\ln(b)$ is strictly increasing and strictly concave in b. Thus the maximand is jointly and strictly concave in (x,b). By Berge's Maximum Theorem, the optimal selections $(x^\star,b^\star)(m,\omega)$ are continuous and unique at any m. Since $c \mapsto c(q)$ is bijective, then

$$q^\star(m,\omega) = c^{-1}\left[x^\star(m,\omega) - k/\mu\left(b^\star(m,\omega)\right)\right]$$

is continuous in m; and so is $\phi^\star(m,\omega)$.

(*Monotonicity of policies.*) The remainder of this proof establishes that the policy functions are increasing. The key idea of the proof is in showing that the choice set is a lattice equipped with a partial order, that the choice set is increasing in m, and, has increasing differences on the choice set, and the slices of the buyer's objective is supermodular in each given direction of his choice set. By Theorem 2.6.2 of Topkis (1998), these properties are sufficient to ensure that the buyer's objective function is supermodular. Together, these properties suffice, by Theorem 2.8.1 of Topkis (1998), for showing that the buyer's optimal policies are increasing functions in m.

1. The function $R\left(\cdot,\cdot,\cdot,\omega\right)$ in (D.2) has *increasing difference* in (x,b,m) and is therefore supermodular:

   Fix an $m \in [k,\bar{m}]$ and $b \in (0,1]$. (The case of $b = 0$ is trivially uninteresting.) It suffices to optimize over the function $\ln\left[R\left(\cdot,b,m,\omega\right)\right]$ in (D.2). Then the optimizer

   $$\tilde{x}(b,m,\omega) \in \left\{\arg\max_{x\in[k,\bar{m}]}\left\{\ln\left[R\left(x,b,m,\omega\right)\right]\right\}\right\}$$

   is unique for each (m,b,ω), since the objective functions is strictly concave.

Next we show how the value of the objective function has increasing differences in (x, b, m), throughout taking the sequence $\{\omega, \omega_{+1}, ...\}$ as fixed. Thus we will now write $R(x, b, m) \equiv R(x, b, m, \omega)$ to temporarily ease the notation. First, the feasible choice set

$$\mathcal{F}_m := \{(x, b, m) : x \in [0, m], b \in [0, 1], m \in [k, \bar{m}]\},$$

is a partially ordered set with relation $\leq$, and it has least-upper and greatest-lower bounds. It is therefore a sublattice in $\mathbb{R}^3_+$. Observe that $\mathcal{F}_m$ is increasing in m. Second, pick any $m' > m$, $b' > b$, and $x' > x$ in $\mathcal{F}_m$:

(a) For fixed x, consider $m' > m$ and $b' > b$. Then, we can write

$$\begin{aligned}
&R(x, b', m') - R(x, b, m) \\
&\quad = \left[u^Q(x, b') - u^Q(x, b)\right] \\
&\qquad + \beta \left[\bar{V}\left(\frac{\omega(m' - x) + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \bar{V}\left(\frac{\omega(m - x) + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right] \\
&\qquad\quad - \beta \left[\bar{V}\left(\frac{\omega m' + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right].
\end{aligned}$$

Observe that the RHS is separable in b and m: The first term on the right, $u^Q(x, b') - u^Q(x, b) < 0$, shows increasing difference in b. Likewise the remainder two difference terms on the RHS show increasing differences in m. Overall $R(x, b, m)$ has increasing differences on the lattice $[0, 1] \times [0, \bar{m}] \ni (b, m)$.

(b) For fixed m, consider $x' > x$ and $b' > b$. Observe that

$$\begin{aligned}
R(x, b, m) - R(x', b, m) &= \left[u^Q(x, b) - u^Q(x', b)\right] \\
&+ \beta \left[\bar{V}\left(\frac{\omega(m - x) + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \bar{V}\left(\frac{\omega(m - x') + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]. \quad \text{(D.3)}
\end{aligned}$$

Now, using the expression (D.3) twice below, we have that

$$
\begin{aligned}
&\left[R\left(x',b',m\right)-R\left(x,b',m\right)\right]-\left[R\left(x',b,m\right)-R\left(x,b,m\right)\right]\\
&=\left[u^{Q}(x',b')-u^{Q}(x,b')\right]\\
&\qquad+\beta\left[\bar{V}\left(\frac{\omega\left(m-x'\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega\left(m-x\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\\
&\qquad-\left[u^{Q}(x',b)-u^{Q}(x,b)\right]\\
&\qquad-\beta\left[\bar{V}\left(\frac{\omega\left(m-x'\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega\left(m-x\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\\
&\qquad\qquad=\left[u^{Q}(x',b')-u^{Q}(x,b')\right]-\left[u^{Q}(x',b)-u^{Q}(x,b)\right]>0,
\end{aligned}
$$

where the last inequality is implied by the fact that $\left(u^{Q}\right)_{12}(x,b)>0$. Therefore $R(x,b,m)$ has increasing differences on the lattice $[0,m]\times[0,1]\ni(x,b)$.

(c) For fixed b, consider $x'>x$ and $m'>m$. Observe that

$$
\begin{aligned}
&\left[R\left(x',b,m'\right)-R\left(x,b,m'\right)\right]-\left[R\left(x',b,m\right)-R\left(x,b,m\right)\right]\\
&=\left[u^{Q}(x',b)-u^{Q}(x,b)\right]\\
&\qquad+\beta\left[\bar{V}\left(\frac{\omega\left(m'-x'\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega\left(m'-x\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\\
&\qquad-\left[u^{Q}(x',b)-u^{Q}(x,b)\right]\\
&\qquad-\beta\left[\bar{V}\left(\frac{\omega\left(m-x'\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega\left(m-x\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\\
&=\beta\left[\bar{V}\left(\frac{\omega\left(m'-x'\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega\left(m'-x\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\\
&\qquad-\beta\left[\bar{V}\left(\frac{\omega\left(m-x'\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)-\bar{V}\left(\frac{m-x+\tau}{1+\tau\omega},\omega_{+1}\right)\right]\geq 0,
\end{aligned}
$$

where the last weak inequality obtains from the property that $\bar{V}(\cdot,\omega_{+1})\in\mathcal{V}[0,\bar{m}]$, and $\bar{V}(\cdot,\omega_{+1})$ is therefore weakly concave. Therefore $R(x,b,m)$ has increasing differences on the lattice $[0,m]\times[0,\bar{m}]\ni(x,m)$.

From parts (1a), (1b), and (1c), we can conclude that the objective function $R(\cdot,\cdot,\cdot,\omega)$ has increasing differences on $\mathcal{F}_m$. This suffices to prove that the objective function $R(\cdot,\cdot,\cdot,\omega)$ is supermodular (see Topkis, 1998, Corollary 2.6.1), since the domain of the function is a direct product of a finite set of chains (partially ordered sets with no unordered pair of elements), and, the objective function is real valued (see Topkis, 1978).

2. Since $R(\cdot,b,m)$ is supermodular, for fixed choice b, the optimizer $\tilde{x}(b,m,\omega)$ is increasing in (b,m), for given ω:

Let $\tilde{x}(b,m,\omega) = \arg\max_{x\in[0,m]} R(x,b,m)$. From part (1a) above, we can deduce that for fixed $\tilde{x}(b,m)$, $\tilde{R}(b,m) \equiv R[\tilde{x}(b,m,\omega), b, m]$ is supermodular on the lattice $[0,1]\times[0,\bar{m}] \ni (b,m)$. Since $R(x,b,m)$ is strictly decreasing in b, then

$$\tilde{R}(b,m) \equiv R[\tilde{x}(b,m,\omega), b, m]$$

is strictly decreasing in b. Observe that for any $m' \geq m$, where $m', m \in [k, \bar{m}]$, we have

$$\begin{aligned} & R(x,b,m') - R(x,b,m) \\ & = \beta\left[\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right] \\ & \qquad - \beta\left[\bar{V}\left(\frac{\omega m'+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \bar{V}\left(\frac{\omega(m'-x)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right] \geq 0, \end{aligned}$$

since $\bar{V}(\cdot,\omega)$ is concave. Since this inequality holds at each fixed pair(x,b), then,

$$\begin{aligned} \tilde{R}(b,m) & \equiv R[\tilde{x}(b,m,\omega), b, m] \\ & \qquad \leq R[\tilde{x}(b,m,\omega), b, m'] \leq R[\tilde{x}(b,m',\omega), b, m'] \equiv \tilde{R}(b,m'). \end{aligned}$$

The last weak inequality obtains because the choice set is increasing in m, and so $\tilde{x}(b,m,\omega)$ is a feasible selection for the more relaxed problem whose value is

$$R[\tilde{x}(b,m',\omega), b, m'] = \max_{x\in[0,m']} R(x,b,m').$$

From these weak inequalities, we can conclude that $\tilde{R}(b,m)$ is increasing in m.

Now we are ready to apply Theorem 2.8.1 of Topkis (1998) to show that $b^\star(m,\omega)$ increases with m: Let

$$b^\star(m,\omega) = \arg\max_{b\in[0,1]} r(b,m)$$

where $r(b,m) = b\cdot\tilde{R}(b,m)$ and $\tilde{R}(b,m) \equiv R(\tilde{x}(b,m,\omega), b, m, \omega)$. Observe the following identity:

$$\begin{aligned} & [r(b',m') - r(b,m')] - [r(b',m) - r(b,m)] = \\ & \qquad b'\left\{\tilde{R}(b',m') - \tilde{R}(b,m') - [\tilde{R}(b',m) - \tilde{R}(b,m)]\right\} \\ & \qquad + (b'-b)[\tilde{R}(b,m') - \tilde{R}(b,m)], \end{aligned}$$

for any $b, b' \in (0,1], m, m' \in [k,\bar{m}]$ where $b' > b$ and $m' > m$. The first term on the RHS is positive, since $b' > 0$ and since $\tilde{R}(b,m)$ is supermodular in (b,m), then Topkis (1998, Theorem 2.6.1) applies, so that $\tilde{R}(b,m)$ has increasing differences on $[0,1] \times [0,\bar{m}]$ (i.e., the terms in the curly braces are positive). Since we have previously established that $\tilde{R}(b,m)$ is increasing in m, and $b' - b > 0$, then the second term on the RHS is also positive. Thus the objective $r(b,m)$ is supermodular on $[0,1] \times [k,\bar{m}] \ni (b,m)$. (Note that the choice set of b does not depend on m.)

Therefore, by Theorem 2.8.1 of Topkis (1998), the optimal selection $b^{\star}(m,\omega)$ is increasing in m. Since $\tilde{x}(b,m,\omega)$ is increasing in (b,m), for given ω, then we can conclude that the optimal payment choice $x^{\star}(m,\omega) = \tilde{x}(b^{\star}(m,\omega),m,\omega)$ is also increasing in m.

3. The decision $q^{\star}(m,\omega)$ is monotone in m:

   We perform a change of decision variables. Denote $a \equiv \varphi + c(q)$, where, $\varphi \equiv m - x$. Then we have a change of the DM buyer's decision variables from (x,q) to (a,q). From (2.10), we can re-write $m - x = a - c(q)$ and $b = \mu^{-1}[k/(m-a)]$. Since $b \in [0,1]$, the domain of a is $[0, m-k]$, and the domain for q is $[0,a]$. The buyer's problem from (2.17) is thus equivalent to writing

$$B(m,\omega) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) = \max_{a\in[0,m-k],q\in[0,a]} \left\{\mu^{-1}\left(\frac{k}{m-a}\right)\left[u(q)\right.\right.$$
$$\left.\left. + \beta\bar{V}\left(\frac{\omega(a-q)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]\right\}. \quad \text{(D.4)}$$

   Recall we take the sequence $(\omega, \omega_{+1}, ...)$ as parametric here. This problem can be broken down into two steps: Fix (a,ω). Find the optimal q for any a, to be denoted by $\tilde{q}(a,\omega)$, and then, find the optimal a given (a,ω), to be denoted by $a^{\star}(m,\omega)$. Then we can deduce the optimal $q^{\star}(m,\omega) \equiv \tilde{q}[a^{\star}(m,\omega), m, \omega]$. We details these steps below:

   (a) For any fixed a and (m,ω), $\tilde{q}(a,\omega)$ induces the value

$$J(a,\omega) = \max_{q\in[0,a]} \left\{u(q) + \beta\bar{V}\left(\frac{\omega(a-q)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right\}. \quad \text{(D.5)}$$

   Observe that q and J do not depend on m, given a fixed a. The objective function on the RHS is clearly supermodular on the lattice $[0, m-k] \times [0,a] \ni (a,q)$. Since the objective function is strictly concave, the selection $\tilde{q}(a,\omega)$ in unique for every a, given ω. Also, the choice set $[0,a]$ increases with a, and, the objective function is increasing. Therefore, respectively by Theorems 2.8.1 (increasing optimal solutions) and 2.7.6 (preservation of supermodularity) of Topkis (1998), we have that $\tilde{q}(a,\omega)$ and $J(a,\omega)$ are increasing in a.

(b) Given best response $\tilde{q}(a,\omega)$, the optimal $a^{\star}(m,\omega)$ choice satisfies

$$a^{\star}(m,\omega) = \arg\max_{a\in[0,m-k]} g(a,m,\omega),$$

where

$$g(a,m,\omega) = \mu^{-1}\left(\frac{k}{m-a}\right)\left[J(a,\omega) - \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right].$$

(Again, note that we have suppressed dependencies on ω_{+1} since this is taken as parametric by the agent, and, in equilibrium ω_{+1} recursively depends on ω.)

Consider the case $J(a,\omega) - \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) \geq 0$. Since $\mu(b)$ is strictly decreasing in b, and $1/\mu(b)$ is strictly convex in b, then $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly increasing in m, strictly decreasing in a, and is strictly supermodular in (a,m). Pick any $a',a \in [0,m-k]$, and any $m',m \in [k,\bar{m}]$, such that $a' > a$ and $m' > m$. We have the identity:

$$\begin{aligned}
&[g(a',m',\omega) - g(a,m',\omega)] - [g(a',m,\omega) - g(a,m,\omega)] = \\
&\left[\mu^{-1}\left(\frac{k}{m'-a'}\right) - \mu^{-1}\left(\frac{k}{m-a'}\right)\right][J(a',\omega) - J(a,\omega)] \\
&+\left[\mu^{-1}\left(\frac{k}{m'-a}\right) - \mu^{-1}\left(\frac{k}{m'-a'}\right)\right] \\
&\quad\times\left[\beta\bar{V}\left(\frac{\omega m'+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \\
&+\left[\mu^{-1}\left(\frac{k}{m'-a'}\right) - \mu^{-1}\left(\frac{k}{m'-a}\right) - \mu^{-1}\left(\frac{k}{m-a'}\right) - \mu^{-1}\left(\frac{k}{m-a}\right)\right] \\
&\qquad\qquad\times\left[J(a,\omega) - \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]
\end{aligned}$$

The first term on the RHS is positive since $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly increasing in m, and we have previously shown that $J(a,\omega)$ is increasing in a. The second term on the RHS is positive since $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly decreasing in a, and, $\tilde{V}(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$. The last term on the RHS is positive since $\mu^{-1}\left(\frac{k}{m-a}\right)$ is supermodular, and therefore its first term in the product shows increasing differences Topkis (1998, Theorem 2.6.1). Its last term in the product is positive under the case we are considering. Therefore the LHS is positive, and this suffices to establish that $g(a,m,\omega)$ is strictly supermodular (Topkis, 1998, Theorem 2.8.1).

Finally, since the choice set $[0,m-k]$ is increasing in m, the solution $a^{\star}(m,\omega)$ is also increasing in m Topkis (1998, Theorem 2.6.1). Since we have established in part (3a) that $\tilde{q}(m,\omega)$ is increasing in a, then, $q^{\star}(m,\omega) \equiv \tilde{q}[a^{\star}(m,\omega),\omega]$ is also increasing in m.

4. The decision $\phi^{\star}(m,\omega)$ is monotone in m:

Similar to the procedure in the last part, we perform a change of decision variables via $a \equiv \varphi + c(q)$, where, $\varphi \equiv m - x$. The domain for φ is $[0, \min\{m,a\}]$. However, an optimal choice under $b > 0$ means that we will have $\varphi < m$ (the end of period residual balance is less than the beginning of period money balance). This is because, if $\varphi = m$ then it must be that $x = 0$, i.e., the buyer pays nothing; but this is not optimal for the buyer if the buyer faces a positive probability of matching $b > 0$. Moreover, $\varphi < a$, if $u'(0)$ is sufficiently large—i.e., the buyer can always increase utility by raising x (thus lowering φ such that $\varphi < a$ attains). Thus the upper bound on φ will never be binding. As such, the buyer's problem from (2.17) can be re-written as

$$B(m,\omega) - \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) = \max_{a\in[0,m-k],\varphi\geq 0}\left\{\mu^{-1}\left(\frac{k}{m-a}\right)\left[u^{C}(a-\varphi)\right.\right.$$
$$\left.\left.+\beta\bar{V}\left(\frac{\omega\varphi+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]\right\}, \quad \text{(D.6)}$$

where $u^{C}(q) := u \circ c^{-1}(q)$, which is continuously differentiable with respect to $q \geq 0$. For fixed $a \in [0, m-k]$, denote the value

$$J(a,\omega) = \max_{\varphi\geq 0}\left\{u^{C}(a-\varphi) + \beta\bar{V}\left(\frac{\omega\varphi+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right\}, \quad \text{(D.7)}$$

and the optimizer,

$$\tilde{\varphi}(a,\omega) = \arg\max_{\varphi\geq 0}\left\{u^{C}(a-\varphi) + \beta\bar{V}\left(\frac{\omega\varphi+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right\}. \quad \text{(D.8)}$$

Denote also $\tilde{q}(a,\omega) = c^{-1}[a - \tilde{\varphi}(a,\omega)]$.

Given $\tilde{\varphi}(a,\omega)$, the optimal choice over a, i.e., $a^{\star}(m,\omega)$, solves

$$B(m,\omega) - \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) =$$
$$\max_{a\in[0,m-k]}\left\{\mu^{-1}\left(\frac{k}{m-a}\right)\left[J(a,\omega) - \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]\right\}.$$

Applying the similar logic in the proof in part 3 on page OA-§.D. 15, we can show that $\tilde{\varphi}(a,\omega)$ is increasing in a; that $a^{\star}(m,\omega)$ is increasing in m, and therefore, $\varphi^{\star}(m,\omega) \equiv$

$\tilde{\varphi}\left[a^{\star}\left(m,\omega\right),\omega\right]$ is increasing in m. Finally, since

$$\phi^{\star}\left(m,\omega\right):=\left[\omega\varphi^{\star}\left(m,\omega\right)+\tau\right]/\left[\omega_{+1}\left(1+\tau\right)\right],$$

which is a linear transform of $\varphi^{\star}\left(m,\omega\right)$, then $\phi^{\star}\left(m,\omega\right)$ is increasing with m, since $\omega/\left[\omega_{+1}\left(1+\tau\right)\right]>0$.

□

D.2 DM buyer value function and first-order conditions

Let us return to the DM buyer's problem re-written as (D.6) in part (4) of the proof of Lemma 3 on page OA-§.D. 10. The buyer's decision problem over $\varphi\equiv m-x$, for any fixed decision $a\equiv\varphi+c(q)$, yields the value $J\left(a,\omega\right)$ as defined in Equation (D.7) of that proof. The following intermediate results says that the value function $J\left(\cdot,\omega\right)$ is differentiable with respect to a and its marginal value can be related to primitives, i.e.:

Lemma 4. *The marginal value of $J\left(\cdot,\omega\right)$ agrees with the flow DM marginal utility with respect to the buyer's payment x,*

$$J_{1}\left(a,\omega\right)=u'\left[\tilde{q}(a,\omega)\right]\equiv\left(u^{Q}\right)_{1}\left[x^{\star}(m,\omega),b^{\star}(m,\omega)\right]>0. \tag{D.9}$$

Proof. Consider the problem described in Equations (D.6) and (D.7). Observe that since $\varphi\equiv a-c\left(q\right)$, then

$$\tilde{\varphi}(a,\omega)=\arg\max_{\varphi\geq0}\left\{u^{C}\left(a-\varphi\right)+\beta\bar{V}\left(\frac{\omega\varphi+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right\} \tag{D.10}$$

is continuous with respect to a: There is some $\delta'>0$, such that for all $\varepsilon\in[0,\delta']$, the choices $\tilde{\varphi}(a+\varepsilon,\omega)$ and $\tilde{\varphi}(a-\varepsilon,\omega)$ exist. Moreover the optimal selection $\tilde{\varphi}(a,\omega)$ is unique since the objective function in (D.10) is strictly concave by virtue of u^{C} being strictly concave and $\bar{V}$ being concave. Denote also $\tilde{q}(a,\omega)=c^{-1}\left[a-\tilde{\varphi}(a,\omega)\right]$, where the choices $\tilde{q}(a+\varepsilon,\omega)$ and $\tilde{q}(a-\varepsilon,\omega)$ also exist, by continuity of c^{-1} in its argument.

To verify (D.9), we can use the perturbed choices, $\tilde{\varphi}(a+\varepsilon,\omega)$ and $\tilde{\varphi}(a-\varepsilon,\omega)$, for evaluating right- and left-derivatives of the functions u^{C}, $\bar{V}$ and J, in order to "sandwich" the derivative function $J_{1}\left(\cdot,\omega\right)$ and arrive at the claimed result. For notational convenience below, we define the following function

$$K_{\omega}\left[a,\tilde{\varphi}(a,\omega)\right]\equiv u^{C}\left(a-\tilde{\varphi}(a,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a,\omega)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right).$$

Consider first the right derivatives: Take $\delta' \searrow 0$ such that for all $\varepsilon \in [0, \delta']$, the choice $\tilde{\varphi}(a+\varepsilon, \omega)$ is affordable for a buyer a. Since $\tilde{\varphi}(\cdot, \omega)$ is an optimal policy satisfying (D.8), then under action $\tilde{\varphi}(a, \omega)$ we must have that

$$\begin{aligned} J(a,\omega) &= u^C\left(a - \tilde{\varphi}(a,\omega)\right) + \beta \bar{V}\left(\frac{\omega\tilde{\varphi}(a,\omega)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \\ &\geq u^C\left(a - \tilde{\varphi}(a+\varepsilon,\omega)\right) + \beta \bar{V}\left(\frac{\omega\tilde{\varphi}(a+\varepsilon,\omega)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \\ \Leftrightarrow J(a,\omega) &= K_\omega\left[a, \tilde{\varphi}(a,\omega)\right] \geq K_\omega\left[a, \tilde{\varphi}(a+\varepsilon,\omega)\right]. \end{aligned}$$

Again, take $\delta' \searrow 0$ such that $\forall \varepsilon \in [0, \delta']$, the choice $\tilde{\varphi}(a, \omega)$ is affordable for buyer $a + \varepsilon$. Since $\tilde{\varphi}(\cdot, \omega)$ is an optimal policy satisfying (D.8), then under $\tilde{\varphi}(a+\varepsilon, \omega)$ we must have that

$$\begin{aligned} J(a+\varepsilon,\omega) &= u^C\left(a+\varepsilon - \tilde{\varphi}(a+\varepsilon,\omega)\right) + \beta \bar{V}\left(\frac{\omega\tilde{\varphi}(a+\varepsilon,\omega)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \\ &\geq u^C\left(a+\varepsilon - \tilde{\varphi}(a,\omega)\right) + \beta \bar{V}\left(\frac{\omega\tilde{\varphi}(a,\omega)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \\ \Leftrightarrow J(a+\varepsilon,\omega) &= K_\omega\left[a+\varepsilon, \tilde{\varphi}(a+\varepsilon,\omega)\right] \geq K_\omega\left[a+\varepsilon, \tilde{\varphi}(a,\omega)\right]. \end{aligned}$$

Re-write the two inequalities above as

$$\begin{aligned} \frac{K_\omega\left[a+\varepsilon, \tilde{\varphi}(a,\omega)\right] - K_\omega\left[a, \tilde{\varphi}(a,\omega)\right]}{\varepsilon} &\leq \frac{J(a+\varepsilon,\omega) - J(a,\omega)}{\varepsilon} \\ &\leq \frac{K_\omega\left[a+\varepsilon, \tilde{\varphi}(a+\varepsilon,\omega)\right] - K_\omega\left[a, \tilde{\varphi}(a+\varepsilon,\omega)\right]}{\varepsilon}. \end{aligned}$$

Since the composite function u^C—and therefore the objective function in (D.7)—is differentiable with respect to a, $J_1(\cdot, \omega)$ clearly exists. Therefore, the right derivative of this value function must agree with its partial derivative: $\lim_{\varepsilon \searrow 0} J(a+\varepsilon, \omega) = J_1(a, \omega)$. Using this fact, the inequalities above imply

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \frac{K_\omega\left[a+\varepsilon, \tilde{\varphi}(a,\omega)\right] - K_\omega\left[a, \tilde{\varphi}(a,\omega)\right]}{\varepsilon} &\leq J_1(a,\omega) \\ &\leq \lim_{\varepsilon \searrow 0} \frac{K_\omega\left[a+\varepsilon, \tilde{\varphi}(a+\varepsilon,\omega)\right] - K_\omega\left[a, \tilde{\varphi}(a+\varepsilon,\omega)\right]}{\varepsilon}. \end{aligned}$$

Moreover, by continuity of $\tilde{\varphi}(\cdot, \omega)$, we have that $\lim_{\varepsilon \searrow 0} \tilde{\varphi}(a+\varepsilon, \omega) = \tilde{\varphi}(a, \omega)$, so the inequalities above collapse to

$$
\begin{aligned}
u'\left[\tilde{q}(a^{+},\omega)\right] &:= \lim_{\varepsilon\searrow 0} \frac{u^{C}\left(a+\varepsilon-\tilde{\varphi}(a,\omega)\right)-u^{C}\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon} \leq J_{1}\left(a,\omega\right) \\
&\leq \lim_{\varepsilon\searrow 0} \frac{u^{C}\left(a+\varepsilon-\tilde{\varphi}(a,\omega)\right)-u^{C}\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon} =: u'\left[\tilde{q}(a^{+},\omega)\right].
\end{aligned}
$$

However, the first and the last term in the inequalities above are identical, and they are the same as the right derivative of u with respect to $q := \tilde{q}(a,\omega)$, i.e., $u'\left[\tilde{q}(a^{+},\omega)\right]$. Thus, it must be that $u'\left[\tilde{q}(a^{+},\omega)\right] = J_{1}\left(a,\omega\right)$.

Using similar arguments as above, we can also consider the left-hand-side perturbation about a, to evaluate $\tilde{\varphi}(a-\varepsilon,\omega)$. It can be shown that

$$
\begin{aligned}
u'\left[\tilde{q}(a^{-},\omega)\right] &:= \lim_{\varepsilon\searrow 0} \frac{u^{C}\left(a-\varepsilon-\tilde{\varphi}(a,\omega)\right)-u^{C}\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon} \leq J_{1}\left(a,\omega\right) \\
&\leq \lim_{\varepsilon\searrow 0} \frac{u^{C}\left(a-\varepsilon-\tilde{\varphi}(a,\omega)\right)-u^{C}\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon} =: u'\left[\tilde{q}(a^{-},\omega)\right],
\end{aligned}
$$

so that $u'\left[\tilde{q}(a^{-},\omega)\right] = J_{1}\left(a,\omega\right)$.

Combining the two arguments above, we have that

$$
u'\left[\tilde{q}(a,\omega)\right] = u'\left[\tilde{q}(a^{+},\omega)\right] = u'\left[\tilde{q}(a^{-},\omega)\right] = J_{1}\left(a,\omega\right) > 0.
$$

Finally, the equivalence $u'\left[\tilde{q}(a,\omega)\right] = \left(u^{Q}\right)_{1}\left[x^{\star}(m,\omega),b^{\star}(m,\omega)\right]$ can be derived using standard calculus, since the composite function $u^{Q} \equiv u \circ Q$ is a known continuously differentiable function in its arguments (x,b). The assumption on u that marginal utility is everywhere positive renders $u'\left[\tilde{q}(a,\omega)\right] > 0$. This completes the proof of the claim. □

Lemma 5. *At any (m,ω), where $m \in [k,\bar{m}]$ and the buyer matching probability is positive $b^{\star}\left(m,\omega\right) > 0$,*

1. the buyer's marginal valuation of money $B_{1}(m,\omega)$ exists if and only if $\bar{V}_{1}\left[\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega\right]$ exists; and
2. $B(m,\omega)$ is strictly increasing in m.

Proof. Lemma 3 implies that $\tilde{q}(a,\omega)$ is increasing in a. Since we have shown that $u'\left[\tilde{q}(a,\omega)\right] = J_{1}\left(a,\omega\right) > 0$, then $J_{1}\left(a,\omega\right)$ is also decreasing in a. Since $J\left(a,\omega\right)$ is clearly increasing in a, then we conclude that it is also concave in a. The term $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly decreasing and strictly concave in a. Therefore the objective function in (D.6) is strictly concave in a. Thus maximizing (D.6) over a yields a unique optimal selection $a^{\star}(m,\omega)$. Moreover, the objective function in (D.6) is continuously differentiable with respect to a; and using (D.9) we can show that $a^{\star}(m,\omega)$ satisfies the first-order condition:[29]

$$J\left(a^{\star}(m,\omega),\omega\right)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right) + u'\left[\tilde{q}\left(a^{\star}(m,\omega),\omega\right)\right]\cdot\frac{k\cdot\mu'\left[b^{\star}\left(m,\omega\right)\right]b^{\star}\left(m,\omega\right)}{\mu\left[b^{\star}\left(m,\omega\right)\right]^{2}}\begin{cases}=0, & \text{if } a^{\star}(m,\omega)<m-k\\ <0, & \text{if } a^{\star}(m,\omega)=m-k\end{cases}. \tag{D.11}$$

Observe that $b^{\star}\left(m,\omega\right)>0$ implies the buyer has more than enough initial balance for purchasing $q^{\star}\left(m,\omega\right)$, i.e.,

$$m-\varphi^{\star}\left(m,\omega\right)>c\left[q^{\star}(m,\omega)\right]+k\implies a(m,\omega)\equiv\varphi^{\star}\left(m,\omega\right)+c\left[q^{\star}(m,\omega)\right]<m-k.$$

Since $a^{\star}(m,\omega)<m-k$, and $a^{\star}(m,\omega)$ is continuous in m, then there is an $\epsilon>0$ such that the following selections are also feasible: $a^{\star}(m+\epsilon,\omega)<m-k$, and, $a^{\star}(m,\omega)<(m-\epsilon)-k$. Define the open ball $\mathbf{N}_{\epsilon}\left(m\right):=\left(m-\epsilon,m+\epsilon\right)$. Note that for any $m'\in\mathbf{N}_{\epsilon}\left(m\right)$, the selection $a^{\star}\left(m',\omega\right)$ is feasible for an agent m; and $a^{\star}\left(m,\omega\right)$ is feasible for agent m'.

Given that $a^{\star}\left(m,\omega\right)$ is optimal for agent m, and since $\varphi^{\star}(m,\omega)=\tilde{\varphi}\left[a^{\star}\left(m,\omega\right)\right]$, then we have the buyer's optimal value as

$$\begin{aligned}B(m,\omega)=\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)+\max_{a\in[0,m-k],\varphi\geq0}\left\{\mu^{-1}\left(\frac{k}{m-a}\right)\right.\\ \left.\times\left[u\circ c^{-1}\left(a-\varphi\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\right\}\\ =F\left(a^{\star}\left(m,\omega\right),m\right)\geq F\left(a^{\star}\left(m+\epsilon,\omega\right),m\right).\end{aligned}$$

where $F\left(a,m\right):=\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)+\mu^{-1}\left(\frac{k}{m-a}\right)\left[J\left(a,\omega\right)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]$. Similarly, for agent $m+\epsilon$, it must be that

$$B\left(m+\epsilon,\omega\right)=F\left(a^{\star}\left(m+\epsilon,\omega\right),m+\epsilon\right)\geq F\left(a^{\star}\left(m,\omega\right),m+\epsilon\right).$$

[29] Note that $b=\mu^{-1}\left(\frac{k}{m-a}\right)$. The term $\mathrm{d}b/\mathrm{d}a=k/\left(m-a\right)^{2}\times1/\mu'\left[b\right]$ can be derived using the implicit function theorem: Define $H(a,b)=k/\left(m-a\right)-\mu\left[b\right]=0$. Then $\mathrm{d}b/\mathrm{d}a=-H_{a}(a,b)/H_{b}(a,b)$, which yields the result. The first-order condition is thus derived as

$$\left[J\left(a^{\star}(m,\omega),\omega\right)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\frac{k}{\left(m-a^{\star}(m,\omega)\right)^{2}}\frac{1}{\mu'\left[b^{\star}\left(m,\omega\right)\right]} + J_{1}\left(a^{\star}(m,\omega),\omega\right)\mu^{-1}\left(\frac{k}{m-a^{\star}(m,\omega)}\right)\begin{cases}=0, & \text{if } a^{\star}(m,\omega)<m-k\\ <0, & \text{if } a^{\star}(m,\omega)=m-k\end{cases}.$$

Moreover, since $k/(m-a)=\mu(b)$, we can write $\mathrm{d}b/\mathrm{d}a=k/\left(m-a\right)^{2}\times1/\mu'\left[b\right]\equiv\left[\mu\left(b\right)\right]^{2}/k\times1/\mu'\left[b\right]$, and using the relation (D.9), the first-order condition can be further simplified to (D.11).

Clearly,

$$\frac{F\left(a^{\star}\left(m,\omega\right),m+\epsilon\right)-F\left(a^{\star}\left(m,\omega\right),m\right)}{\epsilon}\leq\frac{B\left(m+\epsilon,\omega\right)-B\left(m,\omega\right)}{\epsilon}$$
$$\leq\frac{F\left(a^{\star}\left(m+\epsilon,\omega\right),m+\epsilon\right)-F\left(a^{\star}\left(m+\epsilon,\omega\right),m\right)}{\epsilon}.$$

Since $F\left(a,m\right)$ is continuous and concave in a, and, $a^{\star}\left(m,\omega\right)$ is continuous in m, the following limits exist (Rockafellar, 1970, Theorem 24.1, pp.227-228), and the inequality ordering is preserved in the limit:

$$\lim_{\epsilon\searrow 0}\frac{F\left(a^{\star}\left(m,\omega\right),m+\epsilon\right)-F\left(a^{\star}\left(m,\omega\right),m\right)}{\epsilon}\leq\lim_{\epsilon\searrow 0}\frac{B\left(m+\epsilon,\omega\right)-B\left(m,\omega\right)}{\epsilon}$$
$$\leq\lim_{\epsilon\searrow 0}\frac{F\left(a^{\star}\left(m+\epsilon,\omega\right),m+\epsilon\right)-F\left(a^{\star}\left(m+\epsilon,\omega\right),m\right)}{\epsilon}.$$

Since $\lim_{\epsilon\searrow 0}a^{\star}\left(m+\epsilon,\omega\right)=a^{\star}\left(m,\omega\right)$, the inequalities above are equivalent to

$$\begin{aligned}
&b^{\star}\left(m,\omega\right)\left[J_1\left(a^{\star}\left(m,\omega\right),\omega\right)-\frac{\beta}{1+\tau}\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\\
&+\frac{\beta}{1+\tau}\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\\
&\quad\leq B_1\left(m^{+},\omega\right)\\
&\qquad\leq b^{\star}\left(m,\omega\right)\left[J_1\left(a^{\star}\left(m,\omega\right),\omega\right)-\frac{\beta}{1+\tau}\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\\
&\qquad\qquad+\frac{\beta}{1+\tau}\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right),
\end{aligned}$$

where

$$\begin{aligned}
&\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\\
&\quad:=\lim_{\epsilon\searrow 0}\left(1+\tau\right)\left(\frac{\omega_{+1}}{\omega}\right)\left[\bar{V}\left(\frac{\omega\left(m+\epsilon\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]/\epsilon.
\end{aligned}$$

However, observe that the first and the last terms in the inequalities are identical. Thus we must have that the right derivative of $B\left(\cdot,\omega\right)$ satisfies

$$\begin{aligned}
B_1\left(m^{+},\omega\right)=b^{\star}\left(m,\omega\right)&\left[J_1\left(a^{\star}\left(m,\omega\right),\omega\right)-\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\\
&+\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right).
\end{aligned}$$

By a similar process to arrive at the left derivative of $B(\cdot,\omega)$, we have

$$B_1\left(m^-,\omega\right)=b^\star(m,\omega)\left[J_1\left(a^\star(m,\omega),\omega\right)-\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^-+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]$$
$$+\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^-+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right),$$

where

$$\bar{V}_1\left(\frac{\omega m^-+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)$$
$$:=(1+\tau)\left(\frac{\omega_{+1}}{\omega}\right)\lim_{\epsilon\searrow 0}\left\{\frac{1}{\epsilon}\left[\bar{V}\left(\frac{\omega(m-\epsilon)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]\right\}.$$

Using the result from (D.9) in Lemma 4 on page OA-§.D. 18, we can re-write these right- and left-derivative functions, respectively, as

$$B_1\left(m^+,\omega\right)=b^\star(m,\omega)\left(u^Q\right)_1\left[x^\star(m,\omega),b^\star(m,\omega)\right]$$
$$+\frac{\beta\left[1-b^\star(m,\omega)\right]}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^++\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right),\quad\text{(D.12)}$$

and,

$$B_1\left(m^-,\omega\right)=b^\star(m,\omega)\left(u^Q\right)_1\left[x^\star(m,\omega),b^\star(m,\omega)\right]$$
$$+\frac{\beta\left[1-b^\star(m,\omega)\right]}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^-+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right).\quad\text{(D.13)}$$

From (D.12) and (D.13), it is apparent that $B_1(m,\omega)$ exists if and only if the left- and right-derivatives of $\bar{V}(\cdot,\omega_{+1})$ exist and they agree at the continuation state from m, i.e., if

$$\bar{V}_1\left(\frac{\omega m^-+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)=\bar{V}_1\left(\frac{\omega m^++\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)=\bar{V}_1\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right).$$

This proves the first part of the statement in the Lemma.

Since $\bar{V}(\cdot,\omega_{+1})\in\mathcal{V}[0,\bar{m}]$, it is concave and increasing in m, and therefore,

$$\bar{V}_1\left(\frac{\omega m^-+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\geq\bar{V}_1\left(\frac{\omega m^++\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\geq 0.$$

Since we assumed $b^\star(m,\omega)\in(0,1]$, and by Lemma 4, we have

$$J_1\left(a^\star(m,\omega),\omega\right)\equiv\left(u^Q\right)_1\left[x^\star(m,\omega),b^\star(m,\omega)\right]>0,$$

then (D.12) and (D.13) imply that the first-order left and right derivatives of $B(\cdot, \omega_{+1})$ satisfy:

$$B_1(m^-, \omega) \geq B_1(m^+, \omega) \geq b^\star(m, \omega)\left(u^Q\right)_1[x^\star(m, \omega), b^\star(m, \omega)] > 0.$$

From this ordering, we can conclude that if $b^\star(m, \omega) > 0$, the buyer's valuation $B(m, \omega_{+1})$ is *strictly* increasing with his money balance, m. This proves the last part of the statement in the Lemma. $\square$

Lemma 6. *For any (m, ω), where $m \in [k, \bar{m}]$ and the buyer matching probability is positive $b^\star(m, \omega) > 0$, the optimal policy functions $b^\star$ and $x^\star$, respectively, satisfy the first-order conditions (2.18) and (2.19).*

Proof. We want to show that the first order conditions characterizing the optimal policy functions $b^\star$ and $x^\star$, are indeed (2.18) and (2.19). It is immediate that the objective function (2.17) is continuously differentiable with respect to the choice $b \in [0, 1]$. Holding fixed x, if the optimal choice for b is interior, $b^\star(m, \omega) \in (0, 1)$, then it must satisfies the first order condition (2.18) with respect to b:

$$\begin{aligned} u^Q[x^\star(m,\omega), b^\star(m,\omega)] + b^\star(m,\omega)\left(u^Q\right)_2[x^\star(m,\omega), b^\star(m,\omega)] \\ = \beta\left[\bar{V}\left(\phi(m,\omega), \omega_{+1}\right) - \bar{V}\left(\phi^\star(m,\omega), \omega_{+1}\right)\right]. \end{aligned}$$

The first order condition with respect to x is more subtle. We can derive it by first defining one-sided derivatives of $B(\cdot, \omega)$. Assuming beginning-of-next-period residual balance after current DM trade is positive—i.e.,

$$\phi^\star(m, \omega) = \frac{\omega[m - x^\star(m, \omega)] + \tau}{\omega_{+1}(1 + \tau)} > 0. \tag{D.14}$$

Since (D.14) holds, and since we have shown in Lemma 3 that $x^\star(m, \omega)$ and $\phi^\star(m, \omega)$ are continuous in $m \in [k, \bar{m}]$, then

$$(\phi^\star)^+(m, \omega) := \frac{\omega[m + \varepsilon - x^\star(m, \omega)] + \tau}{\omega_{+1}(1 + \tau)},$$

and,

$$(\phi^\star)^-(m, \omega) := \frac{\omega[m - \varepsilon - x^\star(m, \omega)] + \tau}{\omega_{+1}(1 + \tau)},$$

exist and are feasible (or affordable). From (2.17), the DM buyer's one-sided derivatives of $B(\cdot,\omega)$—i.e., its left- or right-marginal valuation of initial money balance—are, respectively,

$$B_1\left(m^+,\omega\right) = \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right) \times \left\{\left[1-b^\star(m,\omega)\right]\bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)},\omega\right) + b^\star(m,\omega)\,\bar{V}_1\left[\left(\phi^\star\right)^+(m,\omega),\omega_{+1}\right]\right\}, \tag{D.15}$$

and,

$$B_1\left(m^-,\omega\right) = \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right) \times \left\{\left[1-b^\star(m,\omega)\right]\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)},\omega\right) + b^\star(m,\omega)\,\bar{V}_1\left[\left(\phi^\star\right)^-(m,\omega),\omega_{+1}\right]\right\}, \tag{D.16}$$

where

$$\bar{V}_1\left(\frac{\omega m^\pm + \tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) := (1+\tau)\left(\frac{\omega_{+1}}{\omega}\right)\lim_{\varepsilon\searrow 0}\left\{\frac{1}{\varepsilon}\left[\bar{V}\left(\frac{\omega(m\pm\epsilon)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega\right)\right]\right\}.$$

From Lemma 5, we have shown by change of variable, that the one-sided derivatives of $B(\cdot,\omega)$ also satisfy (D.15) and (D.16). These are repeated here for convenience as the following equations:

$$B_1\left(m^+,\omega\right) = \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\left[1-b^\star(m,\omega)\right]\bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)},\omega\right) + b^\star(m,\omega)\left(u^Q\right)_1\left[x^\star(m,\omega),b^\star(m,\omega)\right], \tag{D.17}$$

and,

$$B_1\left(m^-,\omega\right) = \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\left[1-b^\star(m,\omega)\right]\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)},\omega\right) + b^\star(m,\omega)\left(u^Q\right)_1\left[x^\star(m,\omega),b^\star(m,\omega)\right]. \tag{D.18}$$

From the last term on the RHS of each of Equations (D.15), (D.16), (D.17), and, (D.18), we have the observation that

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[(\phi^\star)^+(m,\omega),\omega_{+1}\right]=\frac{\beta}{1+\tau\omega}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[(\phi^\star)^-(m,\omega),\omega_{+1}\right]$$
$$=\left(u^Q\right)_1\left[x^\star(m,\omega),b^\star(m,\omega)\right].$$

Since these marginal valuation functions are evaluated at the DM buyer's optimal choice, it must be that $\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[(\phi^\star)^+(m,\omega),\omega_{+1}\right]=\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[(\phi^\star)^-(m,\omega),\omega_{+1}\right]=\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\phi^\star(m,\omega),\omega_{+1}\right]$, and, that this satisfies the first order condition (2.19), which is

$$\left(u^Q\right)_1\left[x^\star(m,\omega),b^\star(m,\omega)\right]=\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\phi^\star(m,\omega),\omega_{+1}\right].$$

□

E Proof of Theorem 3

Proof. First, we show that the value functions listed in the definition of a SME are unique given ω. For given ω, The CM agent's problem in (2.4) clearly defines a self-map $T^{CM}_\omega:\mathcal{V}[0,\bar{m}]\to\mathcal{V}[0,\bar{m}]$, which preserves monotonicity, continuity and concavity (see Theorem 1). However, for fixed ω, the DM buyer's problem in 2.17 defines an operator $T^{DM}_\omega:\mathcal{V}[0,\bar{m}]\to\mathcal{C}[0,\bar{m}]$, where $\mathcal{C}[0,\bar{m}]\supset\mathcal{V}[0,\bar{m}]$ is the set of continuous and increasing functions on the domain $[0,\bar{m}]$. This operator does not preserve concavity. Note that $\bar{V}(\cdot,\omega)\in\mathcal{V}[0,\bar{m}]$ as previously defined. Now we show that the ex-ante problem in (2.6) and (2.8) defines an operator that maps the CM agent's and the DM buyer's value functions, respectively, $W(\cdot,\omega)=T^{DM}_\omega\bar{V}(\cdot,\omega)$ and $B(\cdot,\omega)=T^{DM}_\omega\bar{V}(\cdot,\omega)$, back into the set of continuous, increasing and concave functions: $T_\omega:\mathcal{V}[0,\bar{m}]\to\mathcal{V}[0,\bar{m}]$. Since T^{CM}_ω and T^{DM}_ω are monotone functional operators that satisfy discounting with factor $0<\beta<1$, then the ex-ante problem in (2.6) and (2.8), which defines the composite operator $T_\omega:\mathcal{V}[0,\bar{m}]\to\mathcal{V}[0,\bar{m}]$, clearly preserves these two properties. (The convexification of the graph of T_ω via lotteries in (2.8) preserves concavity of the image of the operator, thus making it a self-map on $\mathcal{V}[0,\bar{m}]$.) It can be shown that $\mathcal{V}[0,\bar{m}]$ is a complete metric space. Thus $T_\omega:\mathcal{V}[0,\bar{m}]\to\mathcal{V}[0,\bar{m}]$ satisfies Blackwell's conditions, and has a unique fixed point, $\bar{V}(\cdot,\omega)=T_\omega\bar{V}(\cdot,\omega)$, by Banach's fixed point theorem.

Second, we verify the following three properties: (1) By Theorem 1 and Theorem 2, the agent's optimal policies are continuous, single-valued and monotone functions. This implies, for fixed ω, that the Markov kernel $P(\mathbf{s},\cdot)$ in the distributional operator (2.21) is a probability measure, and, $P(\cdot,E)$ for all Borel subsets $E\in\mathcal{B}([0,\bar{m}])$ is a measurable function. (2) Since agent's policies are monotone, then $P(\mathbf{s},\cdot)$ is increasing on $[0,\bar{m}]$.

Thus the Markov kernel is a transition probability function. (3) The equilibrium policies clearly dictate that the monotone mixing conditions of Hopenhayn and Prescott (1992) are satisfied: Consider a DM buyer who has zero money balance. With non-zero probability either by pure luck (α) or by choosing a lottery that induces such an outcome, he will enter the CM to work and to accumulate some positive money balance. Likewise, consider an agent, either in the DM or CM with the highest possible initial balance of $\bar{m}$. Again, with non-zero probability, she will decumulate that balance, either by matching and spending that balance down in the DM, or, by working less and consuming more in the CM. These conditions, are sufficient, by Theorem 2 of Hopenhayn and Prescott (1992), for the Markov operator (2.21) to have a unique fixed point—i.e., regardless of an initial distribution of agents, the recursive operation on the initial distribution converges (in the weak* topology) to the same long run distribution G.[30]

Third, the LHS of (2.20), viz. $(1/\omega)$, is clearly continuous in $\omega \in (0 + \infty)$ and is a downward sloping parabola in the interior of $(0, +\infty)$. The market clearing condition (2.20) is continuous on the RHS: (1) The integrand is clearly continuous in m; and, (2) by Theorems (1) and (2), agents policy functions are continuous in m. By Theorem 1 via Equations (2.12) and (2.15) (evaluated at a stationary state $\omega = \omega_{+1}$), demand for real money balance is continuous in ω. Thus, the distribution $G(\cdot;\omega)$ is continuous in ω in the sense of convergence in the weak* topology (Stokey and Lucas, 1989, Theorem 12.13)—i.e., if $\omega_n \to \omega^\star$, then for each $\omega_n \in \{\omega_n\}_{n\in\mathbb{N}}$, the Markov operator (2.21) defines a (weakly) convergent sequence of distributions: $G(\cdot;\omega_n) \to G(\cdot;\omega^\star)$. The RHS is strictly positive valued for all $\omega \in (0 + \infty)$ since agents' policy functions are non-negative valued and $G(\cdot;\omega)$ is a non-degenerate probability measure. Since the RHS is continuous, finite and positive in ω, and the LHS is a downward-sloping parabola with values in $(0 + \infty)$, then there must be at least one intersection point $\omega^\star \in (0 + \infty)$.

The three parts above establish that a SME exists. □

F Algorithm for finding a SME

The following algorithm presumes the more general setting from Section A, which allowed for a new parameter $\alpha \in [0, 1]$. We compute a SME as follows.

1. Fix a guess ω and guess $\bar{V}(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$.

2. Solve for CM policy and value functions:

    - We know $C^\star(m, \omega) = \bar{C}^\star$ already using Equation (2.14).

[30] Alternatively, one could check the more relaxed set of necessary and sufficient conditions of Kamihigashi and Stachurski (2014, Theorem 2) to guarantee that there is a unique distribution for a given ω, in a steady state SME.

- For fixed $\bar{C}^{\star}$, and, given guess of $\bar{V}(\cdot,\omega)$, iterate on Bellman Equation (B.4) solving a one-dimensional (1D) optimization problem over choice $y^{\star}(\cdot,\omega)$.
    - Note: By Equation (2.15), the solution $y^{\star}(\cdot,\omega) = \bar{y}^{*}(\omega)$ should be a constant with respect to m.
- Back out $l^{\star}(m,\omega)$ using the binding budget constraint in Equation (2.16).
- Store value function $W^{\star}(\cdot,\omega)$.

3. Solve for DM policy and value functions:

    - For each $m \leq k$, set
        - $b^{\star}(m,\omega) = x^{\star}(m,\omega) = q^{\star}(m,\omega) = 0$
        - $B(m,\omega) = \beta\bar{V}[\phi(m,\omega),\omega_{+1}]$,

      where $\phi(m,\omega) := (m+\tau)/(1+\tau\omega)$.
    - For each $m \in [k,\bar{m}]$,
        - Invert first-order condition (2.19) to obtain implicit $b[m,x(m,\omega),\omega]$.
        - Plug the implicit expression for $b[m,x(x,\omega),\omega]$ into Bellman Equation (2.17), and do a 1D optimization over choices $x(m,\omega)$.
        - Get optimizer $x^{\star}(m,\omega)$ and corresponding value $B^{\star}(m,\omega)$.
        - Use previous step to now back out $b^{\star}(m,\omega)$.

4. Solve ex-ante decision problem:

    - Given approximants $W^{\star}(m,\omega)$ and $B^{\star}(m,\omega)$, solve the lottery problem (2.6) and (2.8).
    - Get policies $\left\{\pi_1^{j,\star}(m,\omega)\right\}_{j\in J}$ and $\left\{z_1^{j,\star}(m,\omega), z_2^{j,\star}(m,\omega)\right\}_{j\in J}$, where J is endogenous to the solution of (2.6) and (2.8).
    - Get value of the problem (2.6) and (2.8) as $V^{\star}(\cdot,\omega)$.

5. Construct the approximant of the ex-ante value function, $\bar{V}^{\star}(\cdot,\omega) = (1-\alpha)V^{\star}(\cdot,\omega) + \alpha W^{\star}(\cdot,\omega)$.

6. Given policy functions from Steps 2-4, construct limiting distribution $G(\cdot,\omega)$ solving the implicit Markov map (2.21). (See details in Section G on page OA-§.G. 30.)

    - Check if market clearing condition (2.20) holds.
    - If not,
        - generate new guess and set $\omega \leftarrow \omega_{new}$;
        - set $\bar{V}(\cdot,\omega) \leftarrow \bar{V}^{\star}(\cdot,\omega)$; and
        - repeat Steps 2-6 again until (2.20) holds.

Algorithm 1 summarizes the steps above with reference to function names in our actual `Python` implementation. Algorithm 1 is called `SolveSteadyState` in our `Python` class file `cssegmod.py`.

Algorithm 1 Solving for an SME

Require: $\alpha \in [0,1)$, $\omega > 0$, $\bar{V}(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$, $N_{\max} > 0$

for $n \leq N_{\max}$ **do**

    $(W^\star, \bar{C}^\star, l^\star, y^\star) \leftarrow \texttt{WorkerProblem}(\bar{V},\omega)$

    $(B^\star, b^\star, x^\star, q^\star) \leftarrow \texttt{BuyerProblem}(\bar{V},\omega)$

    $\tilde{V} \leftarrow \max\{B^\star(\cdot,\omega), W^\star(\cdot - \chi,\omega)\}$

    $\left(V^\star, \{z^{\star,j}, \pi^{\star,j}\}_{j\in J}\right) \leftarrow \texttt{ConvexHull}\left[\text{graph}\left(\tilde{V}\right)\right]$

    $\bar{V}^\star \leftarrow \alpha W^\star + (1-\alpha)V^\star$

    $\mathbf{v} \leftarrow (\bar{V}, B, W)$

    $\mathbf{p} \leftarrow \langle \{\pi_1^{j,\star}, z_1^{j,\star}, z_2^{j,\star}\}_{j\in J}, (b^\star, x^\star, y^\star, \bar{C}^\star)\rangle$

    $G \leftarrow \texttt{Distribution}(\mathbf{p},\mathbf{v})$

    $\omega^\star \leftarrow \texttt{MarketClearing}(G)$

    $e \leftarrow \max\{\|\bar{V}^\star - \bar{V}\|, \|\omega^\star - \omega\|\}$

    if $e < \varepsilon$ **then**

        STOP

    else

        $(\bar{V},\omega) \leftarrow (\bar{V}^\star,\omega^\star)$

        CONTINUE

    end if

end for

    return $\mathbf{p}, \mathbf{v}, G, \omega^\star$

F.1 A novel computational scheme

We solve for a SME following the pseudocode F. Recall that the directed search problem makes the value function $\tilde{V}(\cdot,\omega)$ non-concave. Since there may exist lotteries that make agents better off than playing pure strategies over participating in DM (as buyer) or CM (as consumer/worker), we have to devise a means for finding these lotteries that convexify the graph of the function $\tilde{V}(\cdot,\omega)$.[31]

An existing way to do this in the literature is to use a grid $\mathrm{M}_g := \{0 < \cdots < \bar{m}\}$ to approximate the function's original domain of $[0,\bar{m}]$. Then, around each finite element of M_g, one must check if there is a linear segment that *approximately* convexifies graph $\left[\tilde{V}(\cdot,\omega)\right]$.[32] This approximation scheme works fine when we only have a lottery where the lower bound in M_g is included, i.e., a lottery on a set like $\{z_1, z_2\}$, where $z_1 = 0$,

[31] Interestingly, there is parallel similarity between our problem here and those in computational dynamic games. In the latter, non-convexities may sometimes arise in equilibrium payoff sets, and convexification of these payoff correspondence images are rationalized through a public randomization (sunspot) device, instead of lotteries or behavior strategies (see, e.g., Kam and Stauber, 2016).

[32] See part (v) of the proof of Theorem 3.5 in Menzio et al. (2013) for an exact theoretical underpinning of this scheme. We thank Amy Sun for sharing her `MATLAB` code for Menzio et al. (2013) which confirms this usage.

and, $z_2 < \bar{m}$. It becomes less accurate when lotteries may exist on upper segments of the function, i.e., lotteries on sets like $\{z'_1, z'_2\}$, where $0 < z'_1 < z'_2 < \bar{m}$, but we have no prior knowlege of what z'_1 is. This is because in practice (on the computer) it is not feasible to implement this check which is meant to be done at every element on the domain $[0, \bar{m}]$, not its approximant M_g. As a result, its implementation on M_g may be prone to introducing non-negligible approximation errors, especially when the mesh of M_g is coarse. Thus, one would have to make M_g very fine, but, this will come at the cost of the overall SME solution time.

Instead, we propose a novel alternative here. We can exploit the property that $\tilde{V}(\cdot, \omega)$ has a bounded and convex domain, so then there exists a smallest convex set that contains $g\tilde{V} := \text{graph}\left[\tilde{V}(\cdot, \omega)\right]$, i.e., $\text{conv}\left(g\tilde{V}\right)$. This set is indeed $\text{graph}\left[\bar{V}(\cdot, \omega)\right]$. We utilize `SciPy`'s interface to the fast `QHULL` algorithm to back out a finite set of coordinates representing the convex hull, i.e., $\text{graph}\left[\bar{V}(\cdot, \omega)\right]$. Given these points, we approximate the function $\bar{V}(\cdot, \omega)$ by interpolation on a chosen continuous basis function. We use the family of linear B-splines available from `SciPy`'s `interpolate` class for this purpose. As a residual of this exercise, we can very quickly and directly determine the entire set of possible lotteries that exists with minimal loss of precision, for any given non-convex/concave function $\tilde{V}(\cdot, \omega)$.[33]

G Monte Carlo simulation of stationary distribution

We use a Monte Carlo method to approximate the steady-state distribution of agents at each fixed value of the aggregate state ω, in the `Distribution` step in Algorithm 1 on page OA-§.F. 29. Again, the following algorithm presumes the more general setting from Section A, which allowed for a new parameter $\alpha \in [0, 1]$.

For any current outcome of an agent named (m, ω) we can evaluate her ex-post optimal choices in either the CM (2.4), or the DM (2.5). The outcomes of the decision at each current state for an agent is summarized in Algorithm 2. In words, these go as follows: First, we must identify where the agent is currently in (DM or CM). Second, we evaluate the corresponding decisions and record the agent's end-of-period money balance as m'. Associated with each realized identity m we would also have a record of the agent's actions in that period, e.g., $y^{\star}(m, \omega)$ and $l^{\star}(m, \omega)$ if the agent was in the CM, or, $x^{\star}(m, \omega)$ and $b^{\star}(m, \omega)$ if she was in the DM.

[33]Detailed comments on how this is done can be found in the method `V` in our `Python` class `cssegmod.py`. We implement our solution in pure `Python` 2.7/3.4 (with OpenMPI parallelization of the agent decision problems on 24 logical cores). We have only tested our solutions on a Dell Precision T7810 workstation (with Intel Xeon E5-2680 v3, 2.50GHz, processors) running on the Centos 7.2 GNU/Linux operating system. In all our experiments, we have monotone convergence towards a unique SME solution, regardless of initial guesses on ω and $\bar{V}(\cdot, \omega)$. The average time taken to find the SME is between 90 to 120 seconds, given our hardware setting.

Algorithm 2 `ExPostDecisions( )`

Require: ω, $(B, W) \leftarrow \mathbf{v}$, $(b^\star, x^\star, y^\star) \leftarrow \mathbf{p}$

if $W(m - \chi, \omega) \geq B(m, \omega)$ **then**
  $m' \leftarrow y^\star(m - \chi, \omega)$
else
  Get $u \sim \mathbf{U}[0, 1]$
  if $u \in [0, b^\star(m, \omega)]$ **then**
    Get $x^\star(m, \omega) > 0$
    Get $b^\star(m, \omega) > 0$
    $m' \leftarrow m - x^\star(m, \omega)$
  else
    $x^\star(m, \omega) \leftarrow 0$
    $b^\star(m, \omega) \leftarrow 0$
    $m' \leftarrow m$
  end if
end if
  return m'

Algorithm 2 is then embedded in Algorithm 3 below, the Monte Carlo approximation of the steady state distribution at ω. We begin, without loss, from an agent who had just accumulated money balances at the end of a CM, and track the evolution of the agent's money balances over the horizon $T \rightarrow +\infty$. Theorem 3 implies that if ω is any candidate equilibrium price, and $G(\cdot, \omega)$ is the unique limiting distribution of agents associated with the candidate equilibrium, then the agent will visit each of all possible states $(m, \omega) \in \text{supp}G(\cdot, \omega)$ with frequency $\mathrm{d}G(m, \omega)$, as $T \rightarrow +\infty$.

Algorithm 3 does the following:

1. Begin with an arbitrary agent m.
2. At the start of each date $t \leq T$:
   (a) The agent realizes the shock $z \sim (\alpha, 1 - \alpha)$.
   (b) Conditional on the shock z, the agent goes to the CM for sure (and costlessly), or, makes the ex-ante lottery decision.
   (c) If the agent has to solve the ex-ante decision problem, then we evaluate the corresponding ex-post decisions of the agent.

The main output of Algorithm 3 is the list m^T, which stores the stochastic realization of an agent's money balances each period. The long run distribution of the sample m^T is used to approximate $G(\cdot, \omega)$. Algorithms 2 and 3 can be found in the `Python` class `cssegmod.py`, respectively, as methods `ExPostDecisions` and `Distribution`.

Note that the function `Distribution` will be called each time we have an updated guess of ω. Because the Monte-Carlo problem is serially dependent, the only way to speed up the evaluations at this point is to compile it to machine code and execute it on the fly. The user will have the option to exploit `Numba` (a `Python` interface to the `LLVM` just-in-time compiler).

Algorithm 3 `Distribution()`

Require: $\mathbf{v} \leftarrow (\bar{V}, B, W)$, $\mathbf{p} \leftarrow \langle \{\pi_1^{j,\star}, z_1^{j,\star}, z_2^{j,\star}\}_{j \in J}, (b^\star, x^\star, y^\star, \bar{C}^\star) \rangle$, T, ω

Get $\phi(m, \omega) \leftarrow \frac{m+\tau}{(1+\tau\omega)(1-\delta)}$
Set $m^T \leftarrow \emptyset$
$m \leftarrow y^\star(0, \omega)$
for $t \leq T$ **do**
    $m \leftarrow \phi(m, \omega)$
    Get $u \sim \mathbf{U}[0,1]$
    if $u \in [0, \alpha]$ **then**
        $m' \leftarrow y^\star(m, \omega)$
    else
        if $\exists j \in J$ and $m \in \left[z_1^{j,\star}(m,\omega), z_2^{j,\star}(m,\omega)\right]$ **then**
            Get $u \sim \mathbf{U}[0,1]$
            if $u \in \left[0, \pi_1^{j,\star}(m,\omega)\right]$ **then**
                $m \leftarrow z_1^{j,\star}(m,\omega)$
            else
                $m \leftarrow z_2^{j,\star}(m,\omega)$
            end if
        end if
        $m' \leftarrow$ `ExPostDecisions`$(m, \omega, \mathbf{p}, \mathbf{v})$
    end if
    Set $m^T \leftarrow m^T \cup \{m\}$
    Set $m \leftarrow m'$
end for
return m^T

H Sample SME outcome for an agent

Figure 15 on page OA-§.H. 34 shows a subsample of an agent's existence, for the baseline economy. Corresponding to the DM/CM patterns of spending, we can also observe the subsample's evolution of money balances, in the panel with its vertical axis labelled m, in Figure 15. Here, we can see that at $t = 0$, the agent has his initial real balance as some m. He decides to be in the DM, succeeds in matching with a trading post, and spends a fraction of the balance to consume some positive q. In the following period $t = 1$, he begins with some positive balance—because of transfer $\tau/(\omega(1+\tau)) > 0$ combined with his residual balance—but this amount land in the lottery region; and so the agents plays the lottery. He realizes the high prize of $z_2 = 0.52$ in $t = 1$, and so his money balance is z_2. He matches and gets to consume $q > 0$. (Hence, the record $q_1, x_1, b_1 > 0$.) A similar event realizes again in $t = 2$, so the agent again gets to consume in the DM. In

$t = 3$, having spent his balance on consuming in the DM the previous period, the agent realizes a low, i.e., $z_1 = 0$, lottery payoff and his initial balance is thus zero. However, the agent is able to borrow against his CM income, and thus decides to take a temporary short asset position of $-\chi$ (although his recorded money balance is $m = z_1 = 0$) and enters the CM to work, repay the entry cost, consume in the CM, and save some money balance.[34] That is why we see a record of -1 for the figure panel labelled "match status" for $t = 3$. Subsequently in $t = 4$, he begins again with positive balance from the last CM trade. At this point, he decides to go shopping in the DM and again, spends it all in one round. he wins the high prize in the lottery, and finds it profitable to pay the fixed cost χ, enters the CM and works.

[34] See our earlier Remark 1 on page 17.

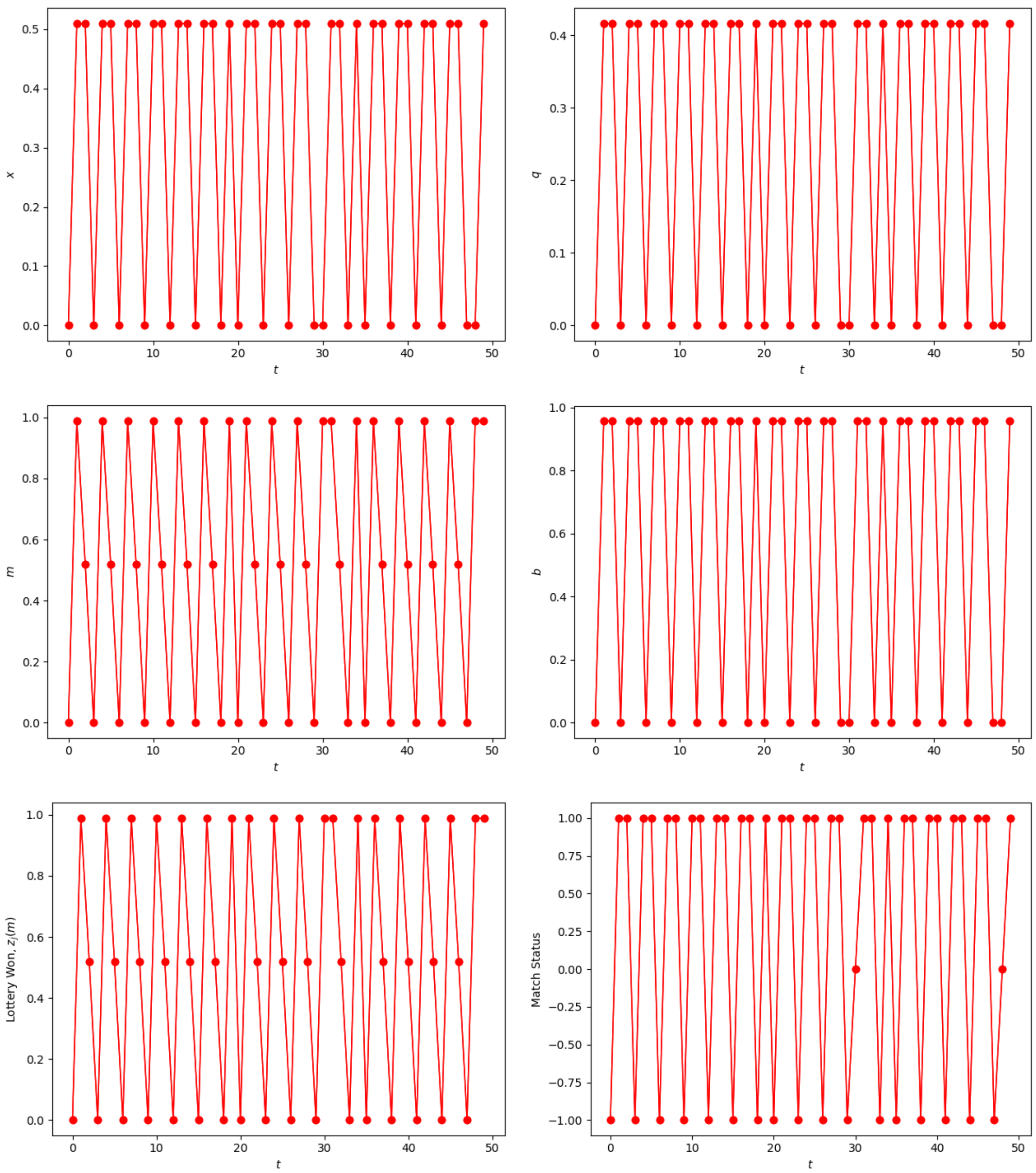

Figure 15: Agent sample path (Benchmark economy). Match Status: 0 (No Match in DM), 1 (Match in DM), -1 (in CM).

In summary, we can observe the following from our simulation: Agents can trade more than once in the DM sometimes. This depends on their luck of the draw in their lottery outcomes. Agents must also pay a fixed cost to enter the CM to load up on money balances. Depending on their money balance, they may sometimes find it worthwhile to borrow against their CM income to pay the fixed cost of CM entry. Thus, we have an equilibrium Baumol-Tobin type of money spending cycle in the model. Since agents endogenously may not have complete consumption insurance, the pattern of consumption in the DM, q in Figure 15, is not completely smooth.

I Robustness and variations on the benchmark economy

This section elaborates in more detail the conclusions made in the paper. Here, we consider two variations or robustness checks on our model assumptions.

First, we show that our insights above are robust to alternative parametrization of the fixed-cost parameter χ.

Second, we consider an extreme assumption that agents face a zero-borrowing constraint when overcoming the fixed cost of CM entry, χ: This alternative economy is tantamount to a reparametrization of the borrowing limit (2.7) in the benchmark setting.

We consider the extreme case of $\chi = 0$ in the last part of this section. This effectively allows agents to participate in the CM costlessly. This experiment illustrates that the key DM intensive-versus-extensive margin trade-off is the main mechanism behind the non-monotone relationship between inflation and money inequality (or consumption inequality).

I.1 Two separate variations: Higher fixed cost and zero-borrowing

Consider first the second environment. The results are qualitatively similar, across increasing inflation rates, to that of the benchmark economy (i.e., the economy with a natural short-sale constraint on overcoming the CM fixed cost). However, for any fixed inflation rate, when one compares this alternative economy with its benchmark counterpart, we have the following additional insights.

τ	Benchmark	Benchmark, $2 \times \chi$	Zero-borrowing Limit
0.000000	0.509568	0.509569	0.525787
0.008394	0.508201	0.508201	0.524484
0.025000	0.505621	0.505621	0.518749

Table 3: Robustness and variations — Mean money holdings

τ	Benchmark	Benchmark, $2 \times \chi$	Zero-borrowing Limit
0.000000	0.711928	0.711885	0.678093
0.008394	0.712490	0.712474	0.678659
0.025000	0.714158	0.714148	0.687622

Table 4: Robustness and variations — DM expected spending relative to holdings

From Table 3 and 4 we can see the following: In the zero-borrowing-limit economy, average money balance is higher, and, equilibrium extensive margin effects in the DM (i.e., on average how fast agents expend their given DM money holdings) are lower than its corresponding benchmark economy.

τ	Benchmark	Benchmark, $2 \times \chi$	Zero-borrowing Limit
0.000000	0.330868	0.330855	0.333409
0.008394	0.331062	0.331057	0.333645
0.025000	0.331440	0.331435	0.333817

Table 5: Robustness and variations — CM participation rate

τ	Benchmark	Benchmark, $2 \times \chi$	Zero-borrowing Limit
0.000000	-0.082472	-0.082479	-0.101590
0.008394	-0.080854	-0.080855	-0.099131
0.025000	-0.071332	-0.071332	-0.063506

Table 6: Robustness and variations — Skewness

However, from Tables 5, 6 and 7, we see that the participation rate in CM is higher, but money distribution is less left-skewed or the Gini index is smaller.

The reason is as follows: In the zero-borrowing economy, agents have a stronger precautionary liquidity-risk insurance motive. Since they cannot borrow to overcome the fixed cost of entering the CM to manage their liquidity needs, then whenever they are in the CM, agents will tend to demand more real balances. Likewise, conditional on being in the DM, agents expect to trade at a lower volume relative to their DM money holdings, as they need to economize on the balance in order to possibly overcome the fixed cost of re-entering the CM. This explains the on-average higher money balance (in comparison to the benchmark economy) and the lower rate of trading in the DM. In return, agents would like to go to the CM more often to demand additional precautionary liquidity. That explains a relatively higher top end of the money distribution relative to the bottom (i.e., a more left-skew distribution), and hence a lower Gini index, in comparison to the benchmark economy's outcome.

A similar reasoning also applies in the first alternative case where we doubled the fixed cost in the benchmark economy.

I.2 Results from limit economy when $x = 0$

The figures below were experiments conducted with the benchmark economy, except for $\chi = 0$. It is clear that in the setting the same qualitative results arises as in the benchmark

τ	Benchmark	Benchmark, $2 \times \chi$	Zero-borrowing Limit
0.000000	0.434360	0.434351	0.409190
0.008394	0.434528	0.434525	0.409407
0.025000	0.435441	0.435439	0.412800

Table 7: Robustness and variations — Gini index

economy discussion in Section 5 in the main paper.

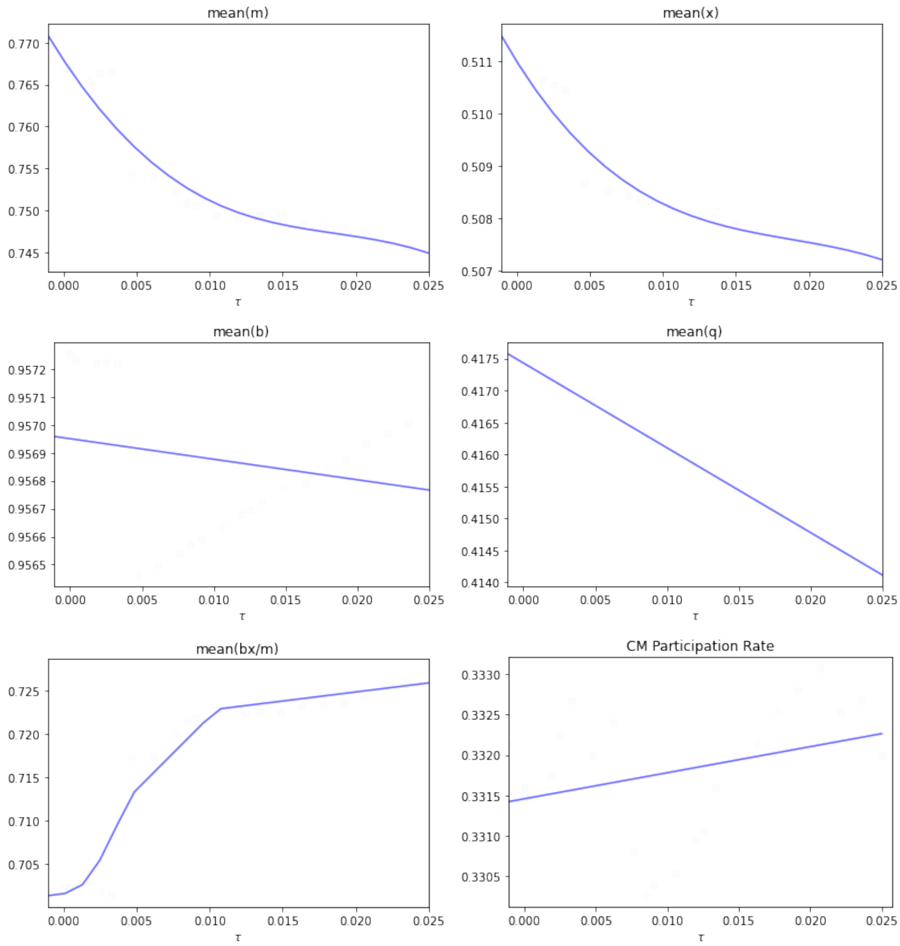

Figure 16: Comparative steady states — mean outcomes (Benchmark but for $\chi = 0$).

Thus, it may have appeared that we complicated the model's mechanism with an additional CM-participation fixed cost parameter χ in the benchmark setting. However, the robustness check shows that the same forces are at work even when $\chi = 0$.

Theoretically, the ocassional CM-participation decision is still present even if $\chi = 0$. Why? As we discussed in the main paper, because the flow preference function of agents is strictly concave, they would like to consume both CM and DM goods in their infinite lifetimes. As a result agents will still transit from CM to DM recurrently, even when it is completely costless to participate in the CM where markets are complete, and even when in the DM, there is a risk that agents may not get to match with trading posts and consume—i.e., they face a liquidity-holdings and insurance risk.

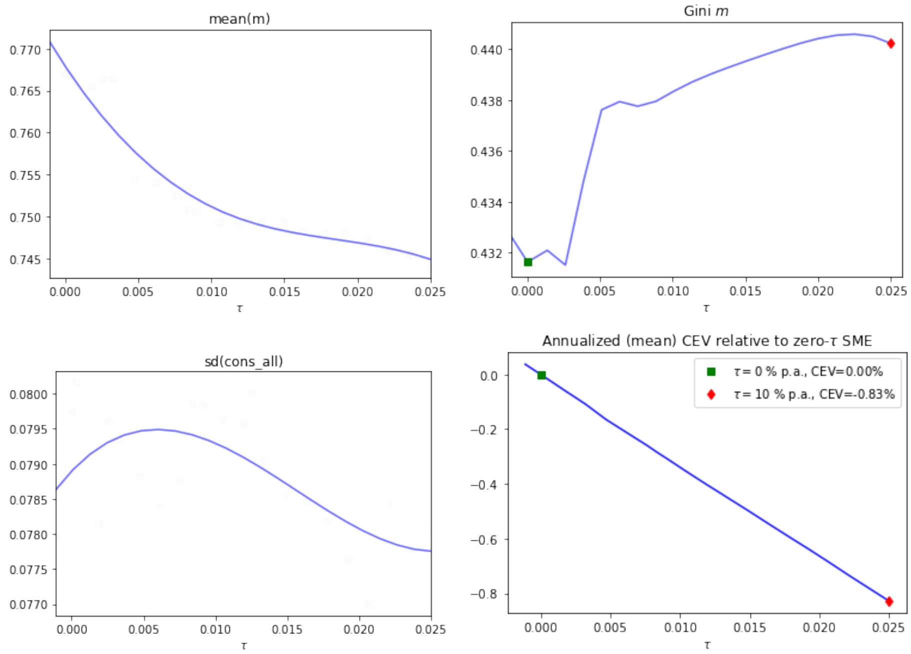

Figure 17: Comparative steady states — distribution (Benchmark but for $\chi = 0$)