



Breaking the Kareken and Wallace Indeterminacy Result

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ANU Working Papers in Economics and Econometrics
613

November, 2013

ISBN: 0 86831 613 X

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Abstract

In this paper we study the endogenous choice to accept fiat objects as media of exchange, the fundamentals that drive their acceptance, and their implications for their bilateral nominal exchange rate. To this end, we consider a small open economy where agents have no restrictions on what divisible fiat currency can be used to settle transactions (i.e. no currency control). We build on [Li, Rocheteau and Weill \(2013\)](#) and allow both fiat currencies to be counterfeited at some fixed costs. The two currencies can coexist, even if one of the currencies is dominated by the other in rate of return. This is driven by an equilibrium outcome in which private information and threats of counterfeiting imposes an equilibrium liquidity constraint on currencies in circulation. Thus, threats of counterfeiting help to pin down a determinate nominal exchange rate, and, to break the Kareken-Wallace indeterminacy result in an environment without ad-hoc currency controls. Finally, we show that with appropriate fiscal policies we can enlarge the set of monetary equilibria with determinate nominal exchange rate.

1 Introduction

When agents have unrestricted access to currency markets and are free to use any currency as the means of payment, [Kareken and Wallace \(1981\)](#) showed that the nominal exchange rate between these currencies is indeterminate.¹ Furthermore, the rate of return on the two currencies must be identical for both of them to circulate. In the last three decades since [Kareken and Wallace \(1981\)](#), no one has been able to generate nominal exchange rate determinacy without imposing an ad hoc friction that inhibits trade using one or more of the currencies. These frictions take many forms ranging from currencies in the utility function, restrictions on the use of currency

*We would like to thank Guillaume Rocheteau, Yiting Li, Ian King, Bruce Preston, Bob King, Charles Engel, Nicolas Jacquet, Ricardo Reis, Cathy Zhang and the participants of the 2013 Midwest Macro Meetings, 8th Workshop on Macroeconomic Dynamics, the 2nd VUW Macro Workshop and the seminar participants at University of Adelaide and University of Queensland. The views expressed are those of the individual authors and do not necessarily reflect official positions of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Board of Governors. Contacts: (P. Gomis-Porqueras) School of Accounting, Economics and Finance, Deakin University, Victoria 3125, Australia. Email: peregomis@gmail.com; (T. Kam) School of Economics, The Australian National University, ACT 0200, Australia. Email: tcy.kam@gmail.com (Corresponding author); (C. Waller) Research Division, Federal Reserve Bank of St. Louis, P.O. Box 442, St. Louis, MO 63166-0442, USA. Email: cwaller@stls.frb.org

¹With perfect currency substitution, there is only one single world market clearing condition determining the supplies and demands of all currencies jointly. Thus an indeterminate monetary equilibrium can only be pinned down by an exogenous selection of the nominal exchange rate. This exogenous information is often interpreted as arbitrary speculation.

for certain transactions, to differential transaction costs. The main contribution of this paper is to break the [Kareken and Wallace \(1981\)](#) indeterminacy result without resorting to any of these types of frictions.

In this paper we study the endogenous choice to accept fiat objects as media of exchange, the fundamentals that drive their acceptance, and their implications for their bilateral nominal exchange rate. To this end, we consider a small open economy version of [Gomis-Porqueras, Kam and Lee \(2013\)](#) where a medium of exchange is essential in the tradition of [Rocheteau and Wright \(2005\)](#) or [Lagos and Wright \(2005\)](#). Agents have no restrictions on what divisible fiat currency can be used to settle transactions nor is there a cost advantage of trading one currency over the other.

What renders two fiat currencies imperfect substitutes is the existence of private information regarding their quality. We build on the insights of [Li, Rocheteau and Weil \(2013\)](#) and allow both fiat currencies to be counterfeited at a fixed cost. The fixed cost of counterfeiting currencies and currency returns can be the same, so that a priori there is no advantage of one currency over the other. Since sellers cannot recognize counterfeit currency, in equilibrium they put an upper-bound limit on how much of each currency they are willing to accept. When neither limit is binding, the nominal exchange rate is indeterminate. However, if the limit is binding for one or both currencies, then we have nominal exchange rate determinacy.

An interesting feature of our results is that there is no counterfeiting in equilibrium. It is the threat of counterfeiting that pins down the nominal exchange rate.² Because of this, both currencies can circulate even though one of them is dominated in rate of return. Finally, we show that when there is nominal exchange indeterminacy there exists a fiscal policy that restores determinacy.

In what follows, Section 2 reviews the literature and Section 3 describes the model environment. In particular, the key private-information friction giving rise to the endogenous liquidity constraints is described in section 3.3 and the equilibrium of an associated signalling game is characterized in section 3.4. The equilibrium characterization of the game is then embedded in an overall general monetary equilibrium in Section 4. In this section, we also consider the implications of the endogenous liquidity constraints for equilibrium and exchange rate determinacy. In section 5, we discuss how cross-country international monetary policies, and, in conjunction with domestic fiscal policy may further rescue the economy from the [Kareken and Wallace \(1981\)](#) indeterminacy result. Finally, we close in Section 6.

2 Related literature

Models in mainstream international monetary economics typically pin down the value of a currency by imposing exogenous assumptions on what objects may be used as media of exchange.

²Central Banks around the world actually spend resources to prevent counterfeiting by incorporating several security features on the notes. Also, counterfeiting currencies is a punishable criminal offence. Several law enforcement entities like INTERPOL, the United States Secret Service and Europol as well as the European Anti-Fraud Office (OLAF), European Central Bank, the US Federal Reserve Bank, and the Central Bank Counterfeit Deterrence Group provide forensic support, operational assistance, and technical databases in order to assist countries in addressing counterfeit currency on a global scale. All these features and efforts substantially reduce the number of circulating counterfeited notes. A fascinating account of the history of counterfeiting can be found in [Mihm \(1993\)](#).

For instance, [Stockman \(1980\)](#) and [Lucas \(1982\)](#), among others, assume that in order to buy a good produced by a particular country, only that country’s currency can be used. That is, in these environments, the demand for a specific fiat currency is solely driven by the demand for goods produced by that particular country. This sort of assumption can be thought as exogenous currency constraint. By construction, this yields determinacy in agents’ portfolio holdings of any two fiat currencies, and therefore determinacy in their nominal exchange rate.³ Other researchers have introduced local currency in the utility function as in [Obstfeld and Rogoff \(1984\)](#) or have differential trading cost advantages through network externalities as in [Uribe \(1997\)](#).

In contrast, exchange rate determinacy in earlier monetary search models is a direct consequence of money being indivisible (see [Matsuyama, Kiyotaki and Matsui, 1993](#); [Waller and Curtis, 2003](#); [Craig and Waller, 2004](#)). The indivisibility of fiat currency creates an automatic capacity constraint on individual choices when deciding their payment instruments. This quantity restriction on fiat money renders imperfect substitutability between competing media of exchange making the nominal exchange rate indeterminate. Similarly, [Head and Shi \(2003\)](#) consider an environment where money is divisible at the aggregate household level, but not at the individual level. Individual buyers are still limited to hold one type of divisible currency at a time. In this sense, models embedding the indivisible-money assumption still capacity constraints on money holdings at a level that matters— individual traders.⁴

The literature has also emphasized differences in the inherent properties of fiat money. For instance, [Camera, Craig and Waller \(2004\)](#) show how agents may optimally hold multiple currencies in equilibrium, and how their spending patterns—i.e. which currency to use first—depend on the relative riskiness of the currencies. The authors model currency risk as random government taxation in a perfect information environment. Here we explore another intrinsic property of fiat currencies, recognizability, and its impact on the determinacy of nominal exchange rates.

The paper closest in spirit to ours is that of [Zhang \(2012\)](#), who considers a recognizability problem between currencies. The informational problem considered by [Zhang \(2012\)](#) is as in [Lester, Postlewaite and Wright \(2012\)](#) where sellers do not accept currencies they cannot recognize. However, the author assumes that local currency is more recognizable than the foreign one, making currencies imperfect substitutes by assumption.

3 Model

We consider an environment where agents can trade domestically and with the rest of the world. In this environment a medium of exchange is essential and agents face private information in some markets. We assume a per-period sequential decentralized-then-centralized market structure and anonymous trading in decentralized markets as in [Lagos and Wright \(2005\)](#) so that a medium of exchange is essential.

³In another strand of literature coined as the “New Open Economy Macroeconomics”, which is partially summarized in [Obstfeld and Rogoff \(1996\)](#) and used extensively for monetary policy prescriptions (see e.g. [Corsetti, Dedola and Leduc, 2010](#)), similar assumptions are in place.

⁴Another alternative search approach, in which media of exchange and goods are all divisible, is also possible. For example, one could also rationalize such strong currency-in-advance constraints as a result of particular (efficient) sequential (take-it-or-leave-it) bargaining games between decentralized traders (see e.g. [Nosal and Rocheteau, 2011](#), section 10.2.2).

General Description The small open economy (SOE) has a continuum of agents of measure 2. Time is discrete and indexed by $t \in \mathbb{N} := \{0, 1, 2, \dots\}$. Each period is divided into two sub-periods with different trading protocols and informational frictions. In the first sub-period, anonymous agents meet pairwise and at random in a decentralized market (DM).⁵ Sellers in this market also face informational asymmetry regarding the quality of the fiat currencies to be exchange for goods. In the second sub-period, all activity occurs in a full information and frictionless centralized market (CM) where agents in the SOE can trade with the rest of the world.

The SOE produces two perishable consumption goods: a non-tradable specialized good produced in DM and an intermediate input that is produced in CM. As in [Rocheteau and Wright \(2005\)](#), DM production is specialized and agents take on fixed trader types so that they are either buyers (consumers) or sellers (producers).⁶ In CM all agents can produce and consume. Agents, however, are restricted to supply their labor within the SOE as migration is not possible.

Preferences Agents derive utility from DM and CM consumption and some disutility from effort. A common discount factor $\beta \in (0, 1)$ applies to utility flows one period ahead. Given the specialization structure in DM where buyers want to consume but can not produce and sellers can produce but do not want to consume, these agents are going to have different preferences. The (discounted) total expected utility of a DM-buyer is given by

$$\mathbb{E} \left\{ \sum_{t=0}^{\infty} \beta^t [u(q_t) + \mathcal{U}(C_t) - H_t] \right\}, \quad (1)$$

where q_t represents DM goods, H_t is the CM labor supply and C_t denotes consumption of composite good which requires domestic ($X_{h,t}$) and foreign ($X_{f,t}$) inputs. Finally, \mathbb{E} is a linear expectations operator with respect to an equilibrium distribution of idiosyncratic agent types.⁷ The utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is such that $u(0) = 0$, $u'(q) > 0$ and $u''(q) < 0$, for all $q \in \mathbb{R}_+$. Also, $\mathcal{U} : \mathbb{R}_+ \rightarrow \mathbb{R}$ has the property that $\mathcal{U}(0) = 0$, $\mathcal{U}'(C) > 0$, and $\mathcal{U}''(C) < 0$, for all $C \in \mathbb{R}_+$. We assume that the CM composite good is given by

$$C_t = D(X_{h,t}, X_{f,t}),$$

where $D : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is a consumption aggregator with the following properties: (i) $D_x(x, y) > 0$, $D_y(x, y) > 0$; (ii) $D_{xx}(x, y) < 0$, $D_{yy}(x, y) < 0$, and $D_{xy}(x, y) > 0$; (iii) $\lim_{x \rightarrow 0} D_x(x, y) =$

⁵The search literature uses the term anonymity to encompass these three frictions: (i) no recordkeeping over individual trading histories (“memory”), (ii) no public communication of histories and (iii) insufficient enforcement (or punishment). An environment with any of these frictions imply that credit between buyer and seller is not incentive compatible.

⁶The justification for this assumption is twofold. First, it allows for a simple description of production specialization (see e.g. [Alchian, 1977](#)). Second, it allows us to abstract away from the additional role of money as a medium of insurance against buyer/seller idiosyncratic shocks. An instance of the latter can be found in the [Lagos and Wright \(2005\)](#) sort of environment.

⁷The model has no aggregate random variables, and therefore, it will turn out that the equilibrium distribution of agent types will depend only on the idiosyncratic random-matching probability $\sigma \in (0, 1)$, and the equilibrium probabilities concerning the acceptability and genuineness of assets in an exchange, respectively, $\pi \in [0, 1]$ and $(\eta, \eta^*) \in [0, 1]^2$. The expected utility setup will be made more precise later in Sections 3.3.3 and 3.3.4.

$\lim_{y \rightarrow 0} D_x(x, y) = +\infty$; and (iv) $\lim_{x \rightarrow +\infty} D_x(x, y) = \lim_{y \rightarrow +\infty} D_x(x, y) = 0$, for any $(x, y) \in \mathbb{R}_+^2$.⁸

The (discounted) total expected utility of a DM-seller is given by

$$\mathbb{E} \left\{ \sum_{t=0}^{\infty} \beta^t [-c(q_t) + \mathcal{U}(C_t) - H_t] \right\}, \quad (2)$$

where the utility cost function $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is such that $c(0) = 0$, $c'(q) > 0$, and $c''(q) \geq 0$. Note that DM-buyers and DM-sellers have identical per-period payoff functions in the CM subperiod given by $\mathcal{U}(C) - H$ as both types of agents can consume and produce in this frictionless market.

Information and Trade Since agents in DM have fixed types and production is specialized, agents face a double coincidence problem. Moreover, since buyers and sellers in DM are anonymous, the only incentive compatible form of payment is fiat money. Buyers and sellers have access to two different and divisible fiat currencies: domestic and foreign. Following [Kareken and Wallace \(1981\)](#), and in contrast to mainstream international macroeconomics, we do not impose any restrictions on which of the currencies, nor the compositions thereof, can be used to settle transactions. However, sellers face asymmetric information, as in [Li, Rocheteau and Weill \(2013\)](#), regarding the quality of the currencies when trading in DM. In the next sections we describe in detail the sub-period trades, and the precise information problem that buyers and sellers are facing will be described in Section 3.3.

3.1 Centralized Market

After trade occurs in DM, agents have access to a frictionless Walrasian international market. In CM agents can trade goods with the rest of the world and rebalance their portfolio of domestic and foreign currencies and decide whether to counterfeit or not.⁹ In particular, before trading in the next DM the buyer has to decide whether to counterfeit these fiat currencies or not in the current CM. As in [Li, Rocheteau and Weill \(2013\)](#), we assume that no fraudulent fiat currencies can be traded for goods in CM as these are detectable by the government who can confiscate and destroy them.

Production Intermediate goods in CM are produced with a linear technology. The total labor supplied in CM is comprised by the effort of DM-buyers (H_t) and DM-sellers (\check{H}_t). This total effort is then transformed into $H_t + \check{H}_t$ units of domestic inputs which are demanded domestically ($X_{h,t}$) and from abroad ($X_{h,t}^*$).

Budget Constraints Each DM-seller faces a sequential budget constraint given by

$$\check{C}_t \leq \check{H}_t - \phi_t(\check{m}_{t+1} - \check{m}_t) - \phi_t e_t(\check{m}_{t+1}^* - \check{m}_t^*), \quad (3)$$

⁸An example of D is the Dixit-Stiglitz CES aggregator used commonly in international trade models.

⁹Labor is not internationally mobile. This assumption is a reasonable approximation of immigration and labor laws for most countries.

where \check{C}_t denotes the seller's consumption of the CM composite good, \check{m}_t and \check{m}_t^* , are her respective initial holdings of genuine domestic and foreign money, e_t represents the current nominal exchange rate which measures how much of the domestic SOE's currency exchanges for one unit of the foreign currency and \check{m}_{t+1} and \check{m}_{t+1}^* are the end-of-period currency portfolio. Finally, ϕ_t (ϕ_t^*) denotes the value of domestic (foreign) currency in units of the domestic (foreign) composite good C_t (C_t^*).

Let (m_t, m_t^*) denote the date- t DM-buyer's beginning-of-CM holdings of home and foreign monies, respectively. Then the corresponding DM-buyer's sequential budget constraint in CM is given by

$$\hat{C}_t \leq H_t - \phi_t(m_{t+1} - m_t) - \phi_t e_t(m_{t+1}^* - m_t^*), \quad (4)$$

where \hat{C}_t denotes the DM-buyer's CM consumption of the composite good, and, (m_{t+1}, m_{t+1}^*) are his end-of-period holdings of home and foreign currencies, respectively.

CM Decisions DM-buyers and DM-sellers choose CM labor, end-of-period currency portfolio and a bundle of domestic and foreign produced intermediate goods. These determine their composite consumption C_t through their respective sequential budget constraints. Note that for every C_t , the optimal decision problem on traded intermediate goods is static. That is, given C_t agents solve a dual expenditure minimization problem given by

$$\min_{X_{h,t}, X_{f,t}} \left\{ P_{h,t} X_{h,t} + e_t P_{f,t}^* X_{f,t} + \phi_t^{-1} [C_t - D(X_{h,t}, X_{f,t})] \right\}; \quad (5)$$

where $P_{h,t}$ ($P_{f,t}^*$) correspond to the price of the domestic (foreign) input in the SOE. This minimization problem can be characterized separately from the agents' dynamic decision problem as in standard international models (see e.g. [Obstfeld and Rogoff, 1996](#)).

In this frictionless international market DM-buyers have the possibility to costly counterfeit both fiat currencies. The cost of counterfeiting is common knowledge and is assumed to be a per-period fixed cost $\kappa > 0$ ($\kappa^* > 0$) for domestic (foreign) currency. However, when trading in DM, sellers are not able to distinguish between genuine and counterfeited monies.

Given the sequential nature of markets in this environment, the DM-buyers' currency portfolio and counterfeiting decisions are dynamic. We will defer the discussion of agents' dynamic decision problem until the next section, and only after we have described the random matching and private information bargaining game between potential DM-buyers and DM-sellers. For now, we note that all DM-buyers will exit each CM with the same currency portfolio after delivering identical labor supply H_t . Likewise, all DM-sellers will exit each CM having supplied the same \check{H}_t . This is due to the quasilinear preference representations for both DM-buyer and DM-seller classes of agents (see [Rocheteau and Wright, 2005](#)).

In the next section we describe the one-sided private information bargaining game between a potentially matched buyer and seller in DM. This problem will span from the end of a period- t CM to the end of a period- $(t+1)$ DM. Then we describe the dynamic decision problems of all agents and describe the monetary equilibrium.

3.2 Decentralized Market

Consider the DM subperiod where trade occurs through random bilateral matches. As in [Gomis-Porqueras, Kam and Lee \(2013\)](#), DM can be interpreted as the non-tradable goods sector where agents do not trade with the rest of the world. Relative to the traditional international literature, here agents trade in a non-Walrasian bilateral matches and agents will be facing asymmetry of information.

Matching There are two fixed types of agents in the DM: buyers (b) and sellers (s). The measures of both b - and s -types are equal to 1. At the beginning of each period $t \in \mathbb{N}$, ex-ante anonymous buyers and sellers enter DM where they are randomly and bilaterally matched. With probability $\sigma \in (0, 1)$ each buyer is pairwise matched with a seller. Since agents are anonymous, exchange supported by contracts that promise repayment in the future is not incentive compatible. Therefore, agents trade with domestic (m) and foreign fiat (m^*) currency.

Feasible-offers Let $\omega := (q, d, d^*)$ denote the terms of trade that specifies how much a seller must produce in DM (q) in exchange for domestic (d) and/or foreign (d^*) fiat currencies. The particulars of the terms of trade ω is an outcome of a bargaining game with private information which we describe below. Denote the set of feasible buyer offers at each aggregate state (ϕ, e) as $\Omega(\phi, e) \ni \omega$.

Given DM preferences and technologies, the corresponding first-best (maximal) quantity traded is $\bar{q} := \bar{q}(\phi, e) \in (0, \infty)$ and satisfies $u'(\bar{q}) = c'(\bar{q})$. For each aggregate state (ϕ, e) , there exists finite and positive numbers $\bar{m} := \bar{m}(\phi, e)$ and $\bar{m}^* := \bar{m}^*(\phi, e)$ solving $(\bar{m} + e\bar{m}^*)\phi = u(\bar{q})$ and $(\bar{m} + e\bar{m}^*)\phi = c(\bar{q})$, since $u(\cdot)$ and $c(\cdot)$ are monotone and continuous functions on every $[0, \bar{q}(\phi, e)]$. That is, the first-best outcomes $(\bar{q}, \bar{m}, \bar{m}^*)(\phi, e)$ will be finite for every (ϕ, e) . Therefore, the set of all feasible offers $\Omega(\phi, e)$ at given (ϕ, e) , is a closed and bounded subset of \mathbb{R}_+^3 , where $\Omega(\phi, e) = [0, \bar{q}(\phi, e)] \times [0, \bar{m}(\phi, e)] \times [0, \bar{m}^*(\phi, e)]$. We summarize this observation in the lemma below.

Lemma 1 *For each given (ϕ, e) , the set of feasible buyer offers $\Omega(\phi, e) \subset \mathbb{R}_+^3$ is compact.*

Having specified the set of all possible offers that the buyer can feasibly make in each state of the economy, we now characterize the private information bargaining game.

3.3 Private Information

The DM-buyers' portfolio composition of genuine and fraudulent fiat currencies is private information as the seller can not distinguish them. The resulting private information problem is modeled as a signaling game between pairs of randomly matched buyers (signal sender) and sellers (signal receiver) as in [Li, Rocheteau and Weill \(2013\)](#). The resulting game is a one-period extensive form game played out in virtual time between each CM and the following period's DM.

A buyer has private information on his accumulation decision and holdings of the two fiat currencies. A matched seller can observe the terms of trade $\omega := (q, d, d^*)$ offered by the buyer but she is not able to distinguish between genuine and counterfeited currencies. In contrast

to standard signaling games, here, signal senders have a choice over their private-information types. These types are defined by the buyer’s portfolio choice at the end of each CM. If the buyer decides to counterfeit fiat currencies she will exchange them for DM goods as in the next CM they are going to be detectable and destroyed. In what follows next, we first describe and characterize the equilibrium of the game.

3.3.1 Endogenous-type Signaling Game

At the beginning of each DM, a seller s is randomly matched with a buyer b . The seller cannot recognize whether the buyer is offering genuine fiat currencies or not. Next we describe the exact timing of events.

Let $\text{CM}(t-1)$ denote the time- $(t-1)$ international frictionless Walrasian market and $\text{DM}(t)$ represent the time- t domestic decentralized and frictional market. One could also think in terms of a $\text{CM}(t)$ and its ensuing $\text{DM}(t+1)$, so the timing notation here does not affect the analysis. For every $t \geq 1$, and given prices, (ϕ_t, e_t) , the timing of the signaling game is as follows:

1. In $\text{CM}(t-1)$ a buyer decides whether or not to costly counterfeit domestic or foreign fiat currency at a one-period fixed cost $\kappa > 0$ and $\kappa^* > 0$, respectively. This decision is captured by the binary action $\chi_j \in \{0, 1\}$ for $j \in J := \{m, m^*\}$, where $\chi_j = 0$ represents “no counterfeiting of currency j ”.
2. The buyer chooses how much domestic $\text{CM}(t-1)$ intermediate good to produce in exchange for genuine currencies, m and/or m^* , and for the composite good C_t .
3. In the subsequent $\text{DM}(t)$, a buyer is randomly matched with a seller with probability σ .¹⁰ Upon a successful match, the buyer makes a take-it-or-leave-it (TIOLI) offer (q, d, d^*) to the seller.¹¹
4. The seller decides whether to accept the offer or not. If the seller accepts, she produces according to the buyer’s TIOLI offer.

The extensive-form game tree of this private information problem is depicted in Figure 1.

[Figure 1 here.]

As in [Li, Rocheteau and Weill \(2013\)](#), this original extensive-form game has the same payoff-equivalent reduced-form game as the following reverse-ordered extensive-form game. Given prices, (ϕ_t, e_t) , we describe the following reverse-ordered game:

1. A DM-buyer signals a TIOLI offer $\omega := (q, d, d^*)$ and commits to ω , before making any (C, H) decisions in $\text{CM}(t-1)$.
2. The buyer decides whether or not to counterfeit the fiat currencies, $\chi_j(\omega) \in \{0, 1\}$, for each $j \in J$.
3. The buyer decides on portfolio $a(\omega) := (m, m^*)(\omega)$ and (C, H) .

¹⁰For simplicity, double-coincidence-of-wants meetings occur with probability zero.

¹¹Implicit in the offer is the buyer signaling that the payment offered consists of genuine assets.

4. The buyer enters $DM(t)$ and Nature randomly matches the buyer with a DM-seller with probability σ .
5. The DM-seller chooses whether to reject or accept the offer, $\alpha(\omega) \in \{0, 1\}$.

This reverse-ordered extensive-form game tree is depicted in Figure 2.

[Figure 2 here.]

This new reverse-ordered game helps refine the set of perfect Bayesian equilibria (PBE) that would arise in the original extensive form game. [In and Wright \(2011\)](#) provide sufficient conditions for the existence of a PBE in an original extensive-form game which is an outcome equivalent to the PBE of its simpler reordered game.¹² [In and Wright \(2011\)](#) call such an equilibrium a *Reordering-invariant Equilibrium* or RI-equilibrium.¹³

3.3.2 Players and Strategies

To simplify exposition, we let x represent x_t , x_{-1} correspond to x_{t-1} , and x_{+1} stand for x_{t+1} , for any date $t \geq 1$. In the next section we characterize the buyer and seller's strategies.

A DM-buyer in $CM(t - 1)$ has individual state, $\mathbf{s}_{-1} := (m_{-1}, m_{-1}^*; \phi_{-1}, e_{-1})$ which is publicly observable in $CM(t - 1)$. Likewise a DM-seller in $CM(t - 1)$ is labelled as $\check{\mathbf{s}}_{-1} := (\check{m}_{-1}, \check{m}_{-1}^*; \phi_{-1}, e_{-1})$. Let $B(\phi, e) := [0, \bar{m}(\phi, e)] \times [0, \bar{m}^*(\phi, e)]$ denote the feasible currency portfolio choice set for a given aggregate state (ϕ, e) .

Definition 2 A pure strategy of a buyer, σ^b , in the counterfeiting game is a triple $\langle \omega, \chi(\omega), a(\omega) \rangle$ comprised by the following:

1. Offer decision rule, $\mathbf{s}_{-1} \mapsto \omega \equiv \omega(\mathbf{s}_{-1}) \in \Omega(\phi, e)$;
2. Binary decision rules on counterfeiting, $\chi_j(\omega) \in \{0, 1\}$, for each $j \in J$; and
3. Asset accumulation decision, $\omega \mapsto a(\omega) \in B(\phi, e)$, and, $(d, d^*) \leq a(\omega)$.

A pure strategy of a seller σ^s is a binary acceptance rule $(\omega, \check{\mathbf{s}}_{-1}) \mapsto \alpha(\omega, \check{\mathbf{s}}_{-1}) \in \{0, 1\}$.

More generally, we allow players to play behavioral strategies given the buyer's posted offer ω . This is the case as quasilinearity in CM makes the buyer's payoff linear in (d, d^*) . This implies that taking a lottery over these payments yields the same utility $u(q)$. Thus, for notational convenience, we drop the lottery over offers when describing a buyer's behavior strategy $\tilde{\sigma}^b$.

¹²See conditions A1-A3 and [\(In and Wright, 2011, Propositions 2 and 3\)](#) for more details.

¹³The characterization of equilibria in [In and Wright \(2011\)](#) is related to the [Cho and Kreps \(1987\)](#) Intuitive Criterion refinement, in the sense that both approaches are implied by the requirement of strategic stability ([Kohlberg and Mertens, 1986](#)). However, the difference in the class of games considered by [In and Wright \(2011\)](#) to that of standard signalling games using [Cho and Kreps \(1987\)](#), is that the class of games considered by the former admits signal senders who have an additional choice of a private-information action. That is, who chooses the private-information type—i.e. Nature in standard signalling games or a Sender in [In and Wright \(2011\)](#)—matters for the game structure. When a strategic and forward-looking Sender can choose his unobserved type, there will be additional ways he can deviate (but these deviations must be unprofitable in equilibrium). Thus standard PBE may still yield too many equilibria in these games with a signalling of private decisions. Further discussions are available in a separate appendix.

Definition 3 A behavior strategy of a buyer $\tilde{\sigma}^b$ is a quadruple $\langle \omega, G[a(\omega)|\omega], \eta(\chi|\omega), \eta^*(\chi|\omega) \rangle$, where

1. $\langle \eta(\cdot|\omega), \eta^*(\cdot|\omega) \rangle$ specifies marginal probability distributions over the $\{0, 1\}$ spaces of each of the two counterfeiting decisions $\chi := (\chi_m, \chi_{m^*})$; and
2. $G(\cdot|\omega)$ is a conditional lottery over each set of feasible asset pairs, $B(\phi, e)$.

A behavior strategy of a seller is $\tilde{\sigma}^s := \pi(\omega)$ which generates a lottery over $\{0, 1\} \ni \alpha$.

Finally, we note that buyers in each $\text{CM}(t-1)$ make the same optimal decisions in subsequent periods. This is the case as agents have CM quasilinear preferences so that history does not matter. Likewise, for the sellers' decisions. All agents, conditional on their DM-buyer or DM-seller types, have the same individual state after they leave CM. Therefore, characterizing the equilibrium of the counterfeiting-bargaining game between a matched anonymous buyer and seller pair in $\text{DM}(t)$ is tractable. Thus, we just can simply focus on the payoffs of any ex-ante DM-buyer and DM-seller.

3.3.3 Buyers' Payoff

Let $W^b(\cdot)$ denote the value function of a DM-buyer at the beginning of $\text{CM}(t)$. Since per-period CM utilities are quasilinear the corresponding CM value function is linear in the buyer's individual state (m, m^*) so that

$$W^b(\mathbf{s}) \equiv W^b(m, m^*; \phi, e) = \phi(m + em^*) + W^b(0, 0; \phi, e). \quad (6)$$

Let us define $Z(C_{-1}; \mathbf{s}_{-1}) = \mathcal{U}(C_{-1}) - C_{-1} + \phi_{-1}(m_{-1} + e_{-1}m_{-1}^*)$ which summarizes the $\text{CM}(t-1)$ flow utility from consuming $(C_{-1}, -H_{-1})$ plus the time- $(t-1)$ real value of accumulating genuine fiat currencies. Then given prices (ϕ, e) and his belief about the seller's behavior $\hat{\pi}$, the DM-buyer's Bernoulli payoff function, $U^b(\cdot)$, can be written as follows:¹⁴

$$\begin{aligned} U^b(C_{-1}, \omega, \eta, \eta^*, G[a(\omega)|\omega], \hat{\pi}|\mathbf{s}_{-1}; \phi, e) = \\ \int_{B(\phi, e)} \left\{ Z(C_{-1}; \mathbf{s}_{-1}) - \phi_{-1}(m + e_{-1}m^*) - \kappa(1 - \eta) - \kappa^*(1 - \eta^*) \right. \\ \left. + \beta\sigma\hat{\pi} \left[u(q) + W^b(m - \eta d, m^* - \eta^* e d^*; \phi, e) \right] \right. \\ \left. + \beta[\sigma(1 - \hat{\pi}) + (1 - \sigma)] W^b(m, m^*; \phi, e) \right\} dG[a(\omega)|\omega]. \end{aligned} \quad (7)$$

Given the linearity of $W^b(\cdot)$, we can further reduce equation (7) to the following expression

$$\begin{aligned} U^b(C_{-1}, \omega, \eta, \eta^*, G[a(\omega)|\omega], \hat{\pi}|\mathbf{s}_{-1}; \phi, e) = -\kappa(1 - \eta) - \kappa^*(1 - \eta^*) \\ + \int_{B(\phi, e)} \left\{ Z(C_{-1}; \mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta \right) \phi e m^* \right. \\ \left. + \beta\sigma\hat{\pi} [u(q) - \phi(\eta d + \eta^* e d^*)] \right\} dG[a(\omega)|\omega] \end{aligned} \quad (8)$$

¹⁴We have imposed symmetry among all sellers for notational simplicity.

which corresponds to the expected total payoff under a given strategy $\tilde{\sigma}^b$ for a DM-buyer in $\text{CM}(t-1)$. Note that the first term of equation (8) is the expected total fixed cost of counterfeiting both currencies. The second term on the right of equation (8) is the utility flow from consuming $(C_{-1}, -H_{-1})$ and the DM(t) continuation value from accumulating currencies in $\text{CM}(t-1)$. The third and fourth term are the expected total cost (equivalently inflation cost) of holding unused currencies between $\text{CM}(t-1)$ and DM(t). The last term is the expected net payoff gain from trades in which the buyer pays for the good q with genuine currencies, with marginal probability measures $\eta(\omega) := (\eta, \eta^*)$, and the seller accepts with probability $\hat{\pi}$, from the buyer's point of view.

Finally, we still have to take into account the buyer's mixed strategy $G(\cdot|\omega)$. In Appendix A.1, we show that in a monetary equilibrium $G(\cdot|\omega)$ is always degenerate, so the buyer's total expected payoff in (8) further simplifies to

$$U^b[C_{-1}, \omega, \eta(\omega), \hat{\pi}|\mathbf{s}_{-1}; \phi, e] = Z(\mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta\right) \phi \eta d - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right) \phi e \eta^* d^* - \kappa(1 - \eta) - \kappa^*(1 - \eta^*) + \beta \sigma \hat{\pi} [u(q) - \phi(\eta d + e \eta^* d^*)]. \quad (9)$$

3.3.4 Sellers' Payoff

A DM-seller's payoff function is simpler. Let $W^s(\cdot)$ denote the seller's value function at the start of any CM. The seller also has a linear value function $W^s(\cdot)$ in currency holdings. Let $Z(\check{C}_{-1}; \check{\mathbf{s}}_{-1}) = \mathcal{U}(\check{C}_{-1}) - \check{C}_{-1} + \phi_{-1}(\check{m}_{-1} + e_{-1}\check{m}_{-1}^*)$ summarize the $\text{CM}(t-1)$ flow utility from consuming $(\check{C}_{-1}, -\check{H}_{-1})$ plus the time- $(t-1)$ real value of accumulating genuine currencies. Note that the DM-seller will always accumulate zero money holdings, because of inflation and the fact that she knows that she has no use of money in the ensuing DM.

Given an offer ω , the seller belief system $\hat{\eta}$ and the seller's response $\pi(\omega)$, her Bernoulli payoff for the game is given by

$$\begin{aligned} U^s(\check{C}_{-1}, \omega, \hat{\eta}, \hat{\eta}^*, \pi(\omega)|\check{\mathbf{s}}_{-1}; \phi, e) &= Z(\check{C}_{-1}; \check{\mathbf{s}}_{-1}) + \beta \sigma \pi(\omega) [-c(q) + W^s(\hat{\eta}d, \hat{\eta}^*d; \phi, e)] \\ &\quad + \beta [\sigma(1 - \pi(\omega)) + (1 - \sigma)] [-c(0) + W^s(0, 0; \phi, e)] \quad (10) \\ &= Z(\check{C}_{-1}; \check{\mathbf{s}}_{-1}) + \beta \sigma \pi(\omega) [\phi(\hat{\eta}d + \hat{\eta}^*ed^*) - c(q)], \end{aligned}$$

where the last equality is a direct consequence of linearity in the seller's CM value function: $W^s(\check{m}, \check{m}^*) = \phi(\check{m} + e\check{m}^*) + W^s(0, 0)$. The last term on the right of the payoff function (10) is the total discounted expected profit arising from the σ -measure of DM(t) exchange, in which the seller accepts an offer ω with probability $\pi(\omega)$ and she anticipates that the buyer pays with genuine assets according to beliefs $(\hat{\eta}, \hat{\eta}^*)$.

3.4 Equilibrium of the Private Information Game

The equilibrium concept for the counterfeiting-bargaining game is Perfect Bayesian in the re-ordered extensive-form game, as in Li, Rocheteau and Weill (2013). More precisely, we utilize the *RI-equilibrium* refinement proposed by In and Wright (2011). In order to solve the game we proceed by backward induction on the game depicted in Figure 2.

3.4.1 Seller's Problem

Let $\hat{\eta}$ be the seller's belief about the buyer's behavior with respect to counterfeiting of fiat currencies. Following a (partially) private buyer history $\langle \omega, \chi(\omega) \rangle$ in which an offer ω is observable and $\chi \sim \hat{\eta}$ is not observable, the seller plays a mixed strategy π to maximize her expected payoff which is given by

$$\pi(\omega) \in \left\{ \arg \max_{\pi' \in [0,1]} \pi' [\phi (\hat{\eta}d + \hat{\eta}^*ed^*) - c(q)] \right\}. \quad (11)$$

3.4.2 Buyer's Counterfeiting Problem

Given history ω and the buyer's belief about the seller's best response, $\hat{\pi}$, the buyer solves the following cost-minimization problem

$$(\eta(\omega), \eta^*(\omega)) = \arg \max_{\eta, \eta^* \in [0,1]} \left\{ -\kappa(1-\eta) - \kappa^*(1-\eta^*) - \left(\frac{\phi-1}{\phi} - \beta \right) \phi m - \left(\frac{\phi-1e-1}{\phi e} - \beta \right) \phi em^* - \beta \sigma \hat{\pi} \phi [\eta + \eta^*ed^*] \right\}. \quad (12)$$

Given that the terms of trade in DM are given by the buyer's TIOLI offer at the beginning of the game, the buyer maximizes her payoff given her conjecture $(\hat{\eta}, \hat{\pi})$ of the continuation play, the buyer commits to an optimal offer $\omega \equiv (q, d_m, d_{m^*})$ which is given by

$$\omega \in \left\{ \arg \max_{\omega' \in \Omega(\phi, e)} \left\{ -\kappa[1-\hat{\eta}] - \kappa^*[1-\hat{\eta}^*] - \left(\frac{\phi-1}{\phi} - \beta \right) \phi \hat{\eta}_m d'_m - \left(\frac{\phi-1e-1}{\phi e} - \beta \right) \phi e \hat{\eta}_{m^*} d'_{m^*} - \left(\frac{\phi-1e-1}{\phi e} - \beta \right) \phi e \hat{\eta}_{m^*} d'_{m^*} - \beta \sigma \hat{\pi} [u(q) - \phi (\hat{\eta}_m d'_m + \hat{\eta}_{m^*} ed'_{m^*})] \right\} \right\} \quad (13)$$

3.4.3 Equilibrium

Having specified the seller's and buyer's respective problems, we can now characterize the resulting equilibrium in the private-information bargaining game.

Definition 4 A reordering-invariant (RI-) equilibrium of the original extensive-form game is a perfect Bayesian equilibrium $\tilde{\sigma} := (\tilde{\sigma}^b, \tilde{\sigma}^s) = \langle \omega, \eta(\omega), \eta^*(\omega), \pi(\omega) \rangle$ of the reordered game such that (11) and (12) are satisfied.

The following proposition provides a simple characterization of a RI-equilibrium in the game.

Proposition 5 (RI-equilibrium) An RI-equilibrium of the counterfeiting-bargaining game is such that

1. Each seller accepts with probability $\hat{\pi} = \pi(\omega) = 1$;
2. Each buyer does not counterfeit: $(\hat{\eta}, \hat{\eta}^*) = (\eta(\omega), \eta^*(\omega)) = (1, 1)$; and

3. Each buyer's TIOLI offer ω is such that:

$$\begin{aligned}
\omega \in \left\{ \arg \max_{\omega \in \Omega(\phi, e)} \left[- \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi d - \left(\frac{\phi_{-1} e_{-1}}{\phi e} - \beta \right) \phi e d^* \right. \right. \\
\left. \left. + \beta \sigma \hat{\pi} [u(q) - \phi (d + e d^*)] \right] \right. \quad s.t. \\
(\zeta) : \quad \phi [d + e d^*] - c(q) = 0, \\
(\nu) : \quad 0 \leq d, \\
(\mu) : \quad d \leq m, \\
(\nu^*) : \quad 0 \leq d^*, \\
(\mu^*) : \quad d^* \leq m^*, \\
(\lambda) : \quad \phi d \leq \frac{\kappa}{\phi_{-1}/\phi - \beta(1 - \sigma)} \equiv \bar{\kappa}(\phi_{-1}/\phi), \\
(\lambda^*) : \quad \phi e d^* \leq \frac{\kappa^*}{\phi_{-1} e_{-1}/\phi e - \beta(1 - \sigma)} \equiv \bar{\kappa}^*(\phi_{-1} e_{-1}/\phi e) \left. \right\}.
\end{aligned} \tag{14}$$

and the RI-equilibrium is unique.

Proof. This can be found in Appendix A.2. ■

As we can see from the RI-equilibrium, ζ represents the Lagrange multiplier associated with the seller's participation constraint, ν (ν^*) is the Lagrangian multiplier corresponding to the non-negativity of the domestic (foreign) payments. Finally, μ (μ^*) represents the feasibility constraint for the local (foreign) fiat money, λ (λ^*) is the Lagrange multiplier corresponding to the liquidity constraint for the local (foreign) fiat money that arise because of the threat of counterfeiting. It is important to highlight that these endogenous liquidity constraints provide an upper bound on the quantities of *genuine* currencies that the seller will accept. These upper bounds depend positively on the fixed cost of counterfeiting and the respective rate of return, and negatively on the degree of matching efficiency σ . Note that a larger σ implies greater matching efficiency in the DM so that buyers and sellers are more likely to meet and trade. This creates a larger incentive for the buyer to produce counterfeits, thus increasing the information problem. Thus, in equilibrium, in order for sellers to accept buyers' offers, each buyer has a tighter upper-bound on his signal/offer of DM payment. A similar intuition applies to the effect of the fixed costs of counterfeiting, and, also to the effect of the aggregate returns on holding *genuine* currencies.

4 Monetary Equilibrium

We can now embed the equilibrium characterization of the game into the overall monetary equilibrium of the model. Since preferences are quasilinear, the infinite history of past games between buyers and sellers does not matter for each current period agents' decision problems. This allows us, as in [Li, Rocheteau and Weill \(2013\)](#), to tractably incorporate the equilibrium characterization of the game previously described, into the overall dynamic general monetary setting. Before we do so, we return to describing the agents' dynamic decision problems.

4.1 Agents' Recursive Problems

DM-buyers' Problem As we previously saw, the beginning-of-CM value function for buyers $W^b(\cdot; \phi, e)$ is linear in the fiat currency portfolio (m, m^*) . As a result, the buyer's intertemporal problem, conditional on an equilibrium of the private-information bargaining game, is given by

$$\max_{C_{-1}, q, d, d^*, m, m^*} \left\{ U^b(C_{-1}, \omega, \eta(\omega), \hat{\pi} | \mathbf{s}_{-1}; \phi, e) \quad \text{s.t.} \right. \quad (15a)$$

$$(\eta(\omega), \eta^*(\omega)) = (1, 1), \quad \hat{\pi} = 1, \quad (15b)$$

$$(\zeta) : \phi [d + ed^*] - c(q) = 0, \quad (15c)$$

$$(\nu) : 0 \leq d, \quad (15d)$$

$$(\mu) : d \leq m, \quad (15e)$$

$$(\nu^*) : 0 \leq d^*, \quad (15f)$$

$$(\mu^*) : d^* \leq m^*, \quad (15g)$$

$$(\lambda) : \phi d \leq \bar{\kappa}, \quad (15h)$$

$$(\lambda^*) : \phi ed^* \leq \bar{\kappa}^* \left. \right\}. \quad (15i)$$

where the DM-buyer's lifetime expected payoff is given by

$$\begin{aligned} U^b(C_{-1}, \omega, \eta(\omega), \eta^*(\omega), \hat{\pi} | \mathbf{s}_{-1}; \phi, e) = & \mathcal{U}(C_{-1}) - C_{-1} + \phi_{-1}(m_{-1} + e_{-1}m_{-1}^*) - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi m \\ & - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta \right) \phi em^* + \beta \sigma [u(q) - \phi (d + ed^*)]. \end{aligned} \quad (15j)$$

In contrast to a full information setting, the threat of counterfeits which is private information to buyers, introduces additional endogenous state-dependent liquidity constraints (15h)-(15i) into a buyer's Bellman equation problem. These endogenous liquidity constraints are going to play an important role in determining the coexistence of the two currencies and the determinacy of nominal exchange rates.

The corresponding first order conditions of the DM-buyers problem are given by

$$1 = \mathcal{U}'(C), \quad (16)$$

$$0 = \beta \sigma u'(q) - \zeta c'(q), \quad (17)$$

$$\beta \sigma = \zeta + \nu - \mu - \lambda, \quad (18)$$

$$\beta \sigma = \zeta + \nu^* - \mu^* - \lambda^*, \quad (19)$$

$$\mu = \frac{\phi_{-1}}{\phi} - \beta, \quad (20)$$

$$\mu^* = \frac{\phi_{-1}^*}{\phi^*} - \beta. \quad (21)$$

$$\zeta \geq 0, \nu \geq 0, \nu^* \geq 0, \mu \geq 0, \mu^* \geq 0, \lambda \geq 0, \lambda^* \geq 0. \quad (22)$$

Note that Equation (16) describes the optimal within-period labor versus consumption trade-

off in CM, where the marginal disutility of labor is -1 and the real-wage (marginal product of labor) is 1. Equation (17) corresponds to the first order condition for DM output which equates the marginal benefit of consuming and marginal value of the payment to the seller. Since the buyer offers a TIOLI, the payment is equal to the seller's DM production cost. Equations (18) and (19) summarize the optimal choice with respect domestic and foreign payment, respectively, and equate the value of holding a particular fiat currency from one CM to the next versus trading it in DM. Finally, equations (20) and (21) describe the optimal accumulation of local and foreign currency which of course depend on its implied rate of return. Equations (18) and (20) (or (18) and (20)) imply a sequence of intertemporal consumption Euler inequalities, where domestic (or foreign) currency is used as store of value.

DM-sellers' Problem A DM-seller's problem, embedding the game's equilibrium, is simpler as sellers cannot counterfeit. This is given by

$$\max_{\check{C}_{-1}} \left\{ U^s(\check{C}_{-1}, \omega, \eta(\omega), \eta^*(\omega), \hat{\pi} | \check{s}_{-1}; \phi, e) \quad \text{s.t.} \quad (\eta(\omega), \eta^*(\omega)) = (1, 1) \quad \text{and} \quad \hat{\pi} = 1 \right\}; \quad (23)$$

where each seller's Bernoulli payoff is given by

$$\begin{aligned} U^s(\check{C}_{-1}, \omega, \eta(\omega), \eta^*(\omega), \hat{\pi} | \check{s}_{-1}; \phi, e) \\ = \mathcal{U}(\check{C}_{-1}) - \check{C}_{-1} + \phi_{-1}(\check{m}_{-1} + e_{-1}\check{m}_{-1}^*) + \beta\sigma [\phi(d + ed^*) - c(q)]. \end{aligned} \quad (24)$$

4.2 Steady State Monetary Equilibrium

Let M and M^* denote the SOE supply of domestic money and the supply of foreign money to the SOE which grow at an exogenous rate Π and Π^* , respectively. Let P_h represent the price level of a unit of domestically produced intermediate CM good, X_h , and $P_f = eP_f^*$ denote the domestic price level of a unit of the imported intermediate good, X_f , where P_f^* is the corresponding world price. The law of one price holds for all CM internationally traded goods as there are no restrictions from goods to be traded internationally.

To simplify the analysis of the equilibrium let us define stationary variables by taking ratios of growing variables as follows:

$$\Pi = \frac{\phi_{-1}}{\phi}; \quad \Pi^* = \frac{\phi_{-1}^*}{\phi^*}; \quad \hat{P}_h = \phi P_h; \quad \hat{P}_f = \phi P_f := \phi e P_f^* \equiv \phi e. \quad (25)$$

Note that since the stationary version of the foreign price is exogenous to the SOE, we normalize $P_f^* \equiv 1$.

Definition 6 *Given initial states (ϕ_0, ϕ_0^*, e_0) and an exogenous path of foreign demand for domestic inputs and inflation $\{X_h^*, \Pi_t^*\}_{t \in \mathbb{N}}$, a monetary equilibrium is a bounded sequence of allocations $\{C_t, q_t, \bar{H}_t, X_{h,t}, X_{f,t}\}_{t \in \mathbb{N}}$, currency portfolios $\{m_t, m_t^*\}_{t \in \mathbb{N}}$, monetary payments $\{d_t, d_t^*\}_{t \in \mathbb{N}}$, and relative prices $\{\hat{P}_{h,t}, \hat{P}_{f,t}, \Pi_t\}_{t \in \mathbb{N}}$, such that for all $t \in \mathbb{N}$:*

1. Agents optimize: (16)-(22), along with optimal demand for home and foreign CM goods:

$$\begin{aligned}\hat{P}_{h,t} &= D_{X_h}(X_{h,t}, X_{f,t}); \\ \hat{P}_{f,t} &= D_{X_f}(X_{h,t}, X_{f,t});\end{aligned}$$

2. CM markets clear:

$$\begin{aligned}(\mathcal{U}')^{-1}(1) &= D(X_{h,t}, X_{f,t}); \\ \bar{H}_t &= X_{h,t} + X_{h,t}^*.\end{aligned}$$

4.3 Steady States

We will focus on steady-state monetary equilibria. We study the implications of the endogenous liquidity constraints for the coexistence of multiple fiat currencies. This also allows us to understand under what conditions there is determinacy of the nominal exchange rate. In a steady state, Definition 6 reduces to the following set of equations:

$$1 = \mathcal{U}'(C); \tag{26}$$

$$\hat{P}_h = D_{X_h}(X_h, X_f), \tag{27}$$

$$\hat{P}_f = D_{X_f}(X_h, X_f), \tag{28}$$

$$1 = \mathcal{U}'(D(X_h, X_f)), \tag{29}$$

$$\bar{H} = X_h + X_h^*, \tag{30}$$

$$0 = \beta\sigma u'(q) - \zeta c'(q), \tag{31}$$

$$\beta\sigma = \zeta + \nu - \mu - \lambda, \tag{32}$$

$$\beta\sigma = \zeta + \nu^* - \mu^* - \lambda^* \tag{33}$$

$$\mu = \Pi - \beta \geq 0, \tag{34}$$

$$\mu^* = \Pi^* - \beta \geq 0; \tag{35}$$

$$c(q) = \phi m + e\phi m^* \Leftrightarrow \zeta > 0; \tag{36}$$

$$\phi m \leq \frac{\kappa}{\Pi - \beta(1 - \sigma)} \Leftrightarrow \lambda \geq 0; \tag{37}$$

$$e\phi m^* \leq \frac{\kappa^*}{\Pi^* - \beta(1 - \sigma)} \Leftrightarrow \lambda^* \geq 0; \tag{38}$$

where Π (Π^*) correspond to the local (foreign) steady state inflation rate.

Some description of this monetary steady state is in order. Equation (26) describes the optimal consumption of the CM composite good. Equation (27) characterizes the demand for the intermediate good, X_h , given its relative price \hat{P}_h (in units of C). Similarly, equation (28) pins down the demand for foreign input X_f , where its relative price in domestic units of C is $\hat{\phi}^*$. Equation (29) gives us the market clearing condition for the good C , which is the output of the aggregator function D . Equation (30) is the market clearing condition for the good X_h produced by the SOE. Notice that all these equations characterize the allocations and prices that relate to trade with the rest of the world. Equation (31) corresponds to the first order condition for DM output. Equations (32) and (33) summarize the optimal choice with respect domestic and

foreign payment, respectively. Equations (34) and (35) describe the optimal accumulation of local and foreign currency. Finally, equations (36), (37) and (38) correspond to the multipliers $\zeta > 0$, $\lambda \geq 0$ and $\lambda^* \geq 0$. For the rest of the paper we focus on equilibria that satisfy $\zeta > 0$ (i.e. DM-seller's participation constraint (36) binds), $\nu = \nu^* = 0$ (i.e. $d, d^* > 0$)—viz. monetary equilibria.

Intuitively, a determinate equilibrium arises if all conditions (26)-(38) hold with strict equality, with the multipliers (μ, μ^*) being strictly positive. That is, there are eleven equations solving for eleven unknowns, in terms of allocations $(C, q, \bar{H}, X_h, X_f, m, m^*)$ and relative prices $(\hat{P}_h, \hat{P}_f, \phi, e)$. However, this may not always hold. In particular, determinacy of the steady state equilibrium, and therefore its nominal exchange rate outcome, depends crucially on cross-country monetary policies (Π, Π^*) . It also depends on the economic structure—in particular the counterfeiting costs (κ, κ^*) and matching friction σ . The following Proposition is the main result of the paper which provides sufficient conditions for the two currencies to coexist, and, for the nominal exchange rate to be determinate even if one of the currencies is dominated in rate of return.

Proposition 7 (Coexistence) *Depending on the relative inflation rates of the two fiat currencies there are three cases to consider.*

1. *When the domestic fiat money is dominated in rate of return ($\Pi > \Pi^*$) and*

- (a) *neither liquidity constraints bind ($\lambda = \lambda^* = 0$), then a unique monetary equilibrium exists with only the low inflation currency circulating;*
- (b) *the foreign liquidity constraint binds ($\lambda^* > \lambda = 0$), then there exists a unique monetary equilibrium where the currencies coexist and the nominal exchange rate is determinate*

$$e = \frac{M^* \bar{\kappa}^*}{M c(q) - \bar{\kappa}^*};$$

- (c) *both liquidity constraints bind ($\lambda^* > \lambda > 0$), then there exists a unique monetary equilibrium where the currencies coexist and the nominal exchange rate is determinate*

$$e = \frac{M^* \bar{\kappa}}{M \bar{\kappa}^*}.$$

2. *When the domestic fiat money dominates in rate of return ($\Pi^* > \Pi$), the coexistence results are the symmetric opposite to those of Case 1.*

3. *When both domestic and foreign currencies have the same rate of return ($\Pi^* = \Pi$) and*

- (a) *neither liquidity constraints bind ($\lambda = \lambda^* = 0$), then the currencies coexist but the individual's currency portfolio and the nominal exchange rate are indeterminate;*
- (b) *both liquidity constraints bind ($\lambda = \lambda^* > 0$), then the currencies coexist and the nominal exchange rate is determinate*

$$e = \frac{M^* \bar{\kappa}}{M \bar{\kappa}^*}.$$

Proof. See Appendix A.3. ■

The key point of this Proposition is that, although there is no counterfeiting in equilibrium, the threat of counterfeiting is all that is required to generate the coexistence and determinacy of the nominal exchange rate. This is true even if the currencies are equivalent in all respects; i.e, the inflation rates and the counterfeiting costs are the same. Thus we have broken the Kareken and Wallace indeterminacy result.

Surprisingly, when the two currencies are identical in every respect, as shown in Case 3b above, the nominal exchange rate is simply the ratio of the two money stocks. This is the exchange rate that is obtained in the standard two currency, cash-in-advance model as in [Stockman \(1980\)](#) and [Lucas \(1982\)](#).

Proposition 7 also contemplates the possibility of just one currency circulating. It is important to highlight that this Proposition describes equilibria that have the property that when the liquidity constraint binds, the marginal value of an additional unit of this currency is zero since the seller will not accept it. At the margin DM-sellers will produce an extra unit of output only for the fiat currency that has a non-binding liquidity constraint. Under these circumstances, the buyer first offers the currency with the best rate of return. Once she hits the endogenous liquidity constraint, the buyer pays for additional units of the DM good with the lower return currency.

In addition to breaking the Kareken and Wallace result, Proposition 7 also implies the following observation.

Proposition 8 (First Best) *When both the domestic and foreign inflation rates converge to the Friedman rule, $\Pi = \Pi^* \rightarrow \beta$, the first best ($u'(q) = c'(q)$) may not be attainable (Case 3b) and the nominal exchange rate may not be determinate (Case 3a).*

Proof. See Appendix A.4. ■

To demonstrate this result, it suffices to provide a counter-example to the claim that the Friedman rule is always optimal. Indeed we show that when there is the threat of counterfeiting, the Friedman Rule may no longer be able to achieve the first best as each DM-seller is not willing to produce more output than what can be afforded by a DM-buyer faced with binding endogenous liquidity constraints.

5 Fiscal Policies

In this section we study how fiscal policies can restore determinacy of the nominal exchange rate in Case 3a. Finding such policies is crucial for policy analysis as an environment with indeterminacy requires the selection of a specific allocation and prices consistent with equilibria. Establishing an appropriate selection rule is extremely difficult.

Let us now consider an environment where the local government is able to impose a tax on all domestic production generated in the centralized market and follows a constant money growth rate rule. For simplicity, we assume that the tax revenues fund wasteful government expenditures. In this new environment the buyer's sequential budget constraint for each CM is

given by

$$C_t \leq (1 - \tau)H_t - \phi_t(m_{t+1} - m_t) - \phi_t e_t(m_{t+1}^* - m_t^*),$$

where τ is the income tax rate. The same taxation assumption applies to the DM-seller's and their corresponding CM budget constraints. This labor tax does not give any of the currencies a distinct advantage over the other. In short, we are not specifying how taxes are paid.

It is straightforward to show that the resulting liquidity constraints for a stationary monetary equilibrium for both domestic and foreign currency are given by

$$\frac{\phi M}{1 - \tau} \leq \frac{\kappa}{\Pi - \beta(1 - \sigma)};$$

$$\frac{e\phi M}{1 - \tau} \leq \frac{\kappa^*}{\Pi^* - \beta(1 - \sigma)}.$$

As we can see from these new endogenous liquidity constraints, production increases the value of fiat currency which reduces the incentives to counterfeit. Note that for a given monetary allocation, we can always find a tax rate $\bar{\tau}$ such that one of the liquidity constraints bind. Thus fiscal policies in coordination with monetary policy can increase the set of equilibria where the nominal exchange rate outcome is determinate. This parallels the analysis of fiscal-monetary policy interconnections for determinacy of equilibria of [Leeper \(1991\)](#) in frictional economies of another kind.

6 Conclusion

In this paper we present a search theoretic model of money with informational asymmetry to study nominal exchange rate determinacy. Agents in this economy trade sequentially in decentralized and Walrasian markets where they can use both currencies, domestic and foreign, to settle transactions. Buyers may counterfeit both fiat currencies at some fixed cost prior to decentralized-market exchanges. Counterfeiting is private information to buyers, as in [Li, Rocheteau and Weill \(2013\)](#), which gives rise to endogenous liquidity constraints on the use of alternative currencies as media of payment.

An interesting feature of our results is that there is no counterfeiting in equilibrium. It is the threat of counterfeiting that pins down the nominal exchange rate. Because of this, both currencies can circulate even though one currency is dominated in the rate of return. Finally, we show that in the case of nominal exchange rate indeterminacy one can find a fiscal policy that will make one of the limits bind, thus restoring equilibrium determinacy. This allows one to rationalize why currencies with similar rates of return remain in circulation (apart from obvious explanations in terms of legal restrictions) as media of exchange.

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A Omitted Proofs

This appendix contains all the proofs of results omitted in the paper.

A.1 Degenerate mixture of asset portfolios

Given that we are interested in monetary equilibria from now on we restrict attention to economies where $\phi_{t-1}/\phi_t \geq \beta$; and $\phi_{t-1}^*/\phi_t^* \equiv e_{t-1}\phi_{t-1}/e_t\phi_t \geq \beta$. The following lemma allows us to simplify the Bernoulli payoff function given by equation (8).

Lemma 9 *Under any optimal measurable strategy $\tilde{\sigma}^b$, genuine portfolio choices are always such that:*

$$m \begin{cases} = \chi_m d_m, & \text{if } \phi_{-1}/\phi > \beta \\ \geq \chi_m d_m, & \text{if } \phi_{-1}/\phi = \beta \end{cases}; \text{ and, } m^* \begin{cases} = \chi_{m^*} d_{m^*}, & \text{if } \phi_{-1}e_{-1}/\phi e > \beta \\ \geq \chi_{m^*} d_{m^*}, & \text{if } \phi_{-1}e_{-1}/\phi e = \beta \end{cases}.$$

Moreover, whenever $\phi_{-1}/\phi = \beta$ (or $\phi_{-1}e_{-1}/\phi e = \beta$), demanding $m > \chi_m d_m$ (or $m^* > \chi_{m^*} d_{m^*}$) is U^b -payoff equivalent for the buyer to demanding $m = \chi_m d_m$ (or $m^* = \chi_{m^*} d_{m^*}$).

Proof. First consider the cases where the returns of either (or both) assets are strictly dominated by β . Then, holding either (or both) assets beyond what is necessary for payments in the DM (i.e. d_m and d_{m^*}) is intertemporally costly since the price levels ϕ^{-1} , and, $(\phi^*)^{-1}$ are respectively growing at the rates $\gamma - 1$ and $\gamma^* - 1$. Thus holding only $m = (1 - \chi_m)d_m$ or (and) $m^* = (1 - \chi_{m^*})d_{m^*}$ is optimal for the DM-buyer under any optimal strategy $\tilde{\sigma}^b$.

Second, consider the cases where the returns of either (or both) assets are equal to β . Then any portfolio demand comprising $m \geq (1 - \chi_m)d_m$ or (and) $m^* \geq (1 - \chi_{m^*})d_{m^*}$ is optimal. However, since W is linear, the only terms involving m and m^* in the buyer's payoff function U^b in (8) are the expected costs of holding unused genuine assets, given by the linear functions

$$\left(\frac{\phi_{-1}}{\phi} - \beta\right) \phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right) \phi e m^*.$$

Observe that in the cases where the asset returns are equal to β , the value of these costs are zero. Therefore, the second statement in the Lemma is true. ■

This result stems from two observations: (i) if the returns on the two fiat currencies are strictly below the discount factor β , then, holding these assets are intertemporally costly; and (ii) if their returns are equal to β , the linearity of $W(\cdot)$ ensures that any excess asset demands beyond what is necessary for trade in the DM is inconsequential to the payoff $U^b(\cdot)$. A.1.

Lemma 9 also implies that each $G(\cdot|\omega)$ consistent with $\tilde{\sigma}^b$ is degenerate, as far as characterizing the Bernoulli payoff function U^b is concerned. That is, given the realization of $\chi(\omega) := (\chi_m(\omega), \chi_{m^*}(\omega))$, we have the following

$$G[a(\omega)|\omega] = \delta_{\{(1-\chi_m)d_m, (1-\chi_{m^*})d_{m^*}\}}, \quad \forall \chi \in \{0, 1\}^2, \quad (39)$$

where δ_E denotes the Dirac delta function defined to be everywhere zero-valued except on events E , on which the function has value 1. In short, we can characterize the buyer's mixed strategy $G(\cdot|\omega)$ (over portfolio accumulation) in the subgame following the buyer's finite history of play, $\langle \omega, \chi(\omega) \rangle$, prior to comprehensively describing equilibrium in the game.

A.2 Proof of Proposition 5

Denote the maximum value of the program in (14), when $\hat{\pi} = \pi(\omega) = 1$ and $(\hat{\eta}, \hat{\eta}^*) = (\eta(\omega), \eta^*(\omega)) = (1, 1)$, as $(U^b)^*$. The aim is to show that an equilibrium $\tilde{\sigma}$ yields the same value as $(U^b)^*$, and it satisfies the characterization in Proposition 5 (Case 1); and that any other candidate strategy $\tilde{\sigma}' := \langle \omega', \eta', \pi' \rangle$ such that $\hat{\pi}' = \pi'(\omega) \neq 1$ and/or $\hat{\eta}' = \eta'(\omega) \neq (1, 1)$ will

induce a buyer's valuation that is strictly less than $(U^b)^*$, and therefore cannot constitute an equilibrium (Cases 2-5).

Consider the subgame following offer ω . Let $\rho(\chi_m, \chi_{m^*})$ denote the joint probability measure on events $\{(\chi_m, \chi_{m^*})\}$, where the pure actions over counterfeiting are $(\chi_m, \chi_{m^*}) \in \{0, 1\}^2$. Denote $\Sigma := 2^{\{0,1\}^2}$ as the power set of $\{0, 1\}^2$. By the definition of probability measures, it must be that $\sum_{z \in \Sigma} \rho(z) = 1$.

The seller's problem in (11) is equivalent to:

$$\pi(\omega) \in \left\{ \arg \max_{\pi' \in [0,1]} \pi' \left[\phi \left([1 - \hat{\rho}(1, 0) - \hat{\rho}(1, 1)]d + [1 - \hat{\rho}(0, 1) - \hat{\rho}(1, 1)]ed^* \right) - c(q) \right] \right\}. \quad (40)$$

This is a linear programming problem in π , given the seller's rational belief system $\hat{\rho}$ and buyer's offer ω . Thus the seller's best response satisfies:

$$\begin{aligned} & \left(\phi([1 - \hat{\rho}(1, 0) - \hat{\rho}(1, 1)]d + [1 - \hat{\rho}(0, 1) - \hat{\rho}(1, 1)]ed^*) - c(q) \right) \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases} \\ \Rightarrow & \left(\pi(\omega) \begin{cases} = 1 \\ = 0 \\ \in [0, 1] \end{cases} \right). \end{aligned} \quad (41)$$

Let $U_{\{z\}}^b \equiv U^b[\omega, \{z\}, \hat{\pi} | \mathbf{s}_{-1}, \phi, e]$ denote the buyer's expected payoff from *realizing* pure actions (χ_m, χ_{m^*}) , given offer ω and rational belief system $\hat{\pi} \in [0, 1]$, where $\{z\} \in 2^{\{0,1\} \times \{0,1\}}$. We have the following possible payoffs following each event $\{z\}$:

$$U_{\{(0,0)\}}^b = - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi d - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta \right) \phi ed^* + \beta \sigma \hat{\pi} [u(q) - \phi(d + ed^*)]; \quad (42)$$

$$U_{\{(0,1)\}}^b = -\kappa^* - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi d + \beta \sigma \hat{\pi} [u(q) - \phi d]; \quad (43)$$

$$U_{\{(1,0)\}}^b = -\kappa - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta \right) \phi ed^* + \beta \sigma \hat{\pi} [u(q) - \phi ed^*]; \quad (44)$$

$$U_{\{(1,1)\}}^b = -\kappa^* - \kappa + \beta \sigma \hat{\pi} u(q). \quad (45)$$

Observe that

$$U_{\{(0,1)\}}^b + U_{\{(1,0)\}}^b = U_{\{(0,0)\}}^b + U_{\{(1,1)\}}^b. \quad (46)$$

There are five cases to consider.

Case 1. Suppose there is a set of candidate equilibria such that $\rho(0, 0) = 1$ and $\rho(z) = 0$, for all $\{z\} \in 2^{\{0,1\} \times \{0,1\}}$ and $z \neq (0, 0)$. Then, $U_{\{(0,0)\}}^b > \max\{U_{\{(1,0)\}}^b, U_{\{(0,1)\}}^b, U_{\{(1,1)\}}^b\}$. Since

$U_{\{(0,0)\}}^b > U_{\{(1,0)\}}^b$ and $U_{\{(0,0)\}}^b > U_{\{(0,1)\}}^b$, then, from (42)-(45) we can derive that

$$\phi d < \frac{\kappa}{\frac{\phi_{-1}}{\phi} - \beta(1 - \sigma \hat{\pi})}, \quad (47)$$

and,

$$\phi e d^* < \frac{\kappa^*}{\frac{\phi_{-1}e_{-1}}{\phi e} - \beta(1 - \sigma \hat{\pi})}. \quad (48)$$

The interpretation from (47) and (48) is that the liquidity constraints on either monies are slack. Therefore the buyer's expected payoff in this case can be evaluated from (42). If $\hat{\pi} < 1$, then from the seller's decision rule (41) we can deduce $\omega \equiv (q, d, d^*)$ must be such that the seller's participation/incentive constraint binds:

$$c(q) = \phi(d + e d^*). \quad (49)$$

Since (49) holds, all we need to do is verify the buyer's payoff. Since, the buyer's liquidity constraints (47) and (48) do not bind at $\hat{\pi} < 1$, a small increment in either payment offered, d or d^* , relaxes (49) and this raises $\hat{\pi}$, and thus the buyer's payoff (42). The maximal payoff to the buyer, keeping the seller in participation, is when $\pi(\omega) = \hat{\pi} = 1$, and the offer $\bar{\omega}$ is such that

$$\begin{aligned} \bar{U}^b \equiv U_{\{(0,0)\}}^b[\bar{\omega} | \pi(\omega) = \hat{\pi} = 1] &= \sup_{\omega} \left\{ U_{\{(0,0)\}}^b[\omega | \pi(\omega) = \hat{\pi} = 1] : \right. \\ &\quad \phi d \leq \frac{\kappa}{\frac{\phi_{-1}}{\phi} - \beta(1 - \sigma)}, \\ &\quad \phi e d^* \leq \frac{\kappa^*}{\frac{\phi_{-1}e_{-1}}{\phi e} - \beta(1 - \sigma)}, \\ &\quad \left. c(q) \leq \phi(d + e d^*) \right\}. \end{aligned}$$

Then it is easily verified that this maximal value coincides with the maximum value of the program given in (14) in Proposition 5, i.e. $\bar{U}^b = (U^b)^*$, since the payoff function is continuous, and the constraints also define a nonempty, compact subset of the feasible set $\Omega(\phi, e) \ni \bar{\omega}$. Since the seller has no incentive to deviate from $\pi(\bar{\omega}) = 1$, then a behavior strategy $\tilde{\sigma} = \langle \bar{\omega}, (1, 1), 1 \rangle$ inducing the TIOLI payoff \bar{U}^b is a PBE.

Case 2. Note that in any equilibrium, a seller will never accept an offer if $\rho(1, 1) = 1$, and, a buyer will never counterfeit both assets with probability 1—counterfeiting for sure costs $\kappa + \kappa^*$ and the buyer gains nothing. Therefore, $\rho(1, 1) < 1$ is a necessary condition for an equilibrium in the subgame following ω . Likewise, all unions of disjoint events with this event of counterfeiting all assets—i.e. $\{(\chi_m, \chi_{m^*})\} \in \{(0, 1)\} \cup \{(1, 1)\}$ or $\{(\chi_m, \chi_{m^*})\} \in \{(1, 0)\} \cup \{(1, 1)\}$ —such that $\rho(0, 1) + \rho(1, 1) = 1$ or $\rho(1, 0) + \rho(1, 1) = 1$, respectively, cannot be on any equilibrium path.

Case 3. Suppose instead we have equilibria in which $\rho(0, 0) + \rho(1, 0) = 1$, $\rho(1, 0) \neq 0$, and $\rho(1, 1) + \rho(0, 1) = 0$, so $U_{\{(1,0)\}}^b = U_{\{(0,0)\}}^b > \max\{U_{\{(0,1)\}}^b, U_{\{(1,1)\}}^b\}$.

Given this case, and from (46), we have $U_{\{(0,1)\}}^b = U_{\{(1,1)\}}^b$. From $U_{\{(1,0)\}}^b = U_{\{(0,0)\}}^b$, and (42)

and (44), respectively, we have:

$$\hat{\pi} = \frac{\kappa - (\phi_{-1}/\phi - \beta)\phi d}{\beta\sigma\phi d}, \quad (50)$$

and,

$$\phi ed^* < \frac{\kappa^*}{\frac{\phi_{-1}e_{-1}}{\phi e} - \beta(1 - \sigma\hat{\pi})}. \quad (51)$$

If $\hat{\pi} < 1$, then from the seller's decision rule (41) we can deduce $\omega \equiv (q, d, d^*)$ must be such that the seller's participation/incentive constraint binds:

$$\begin{aligned} c(q) &= \phi[(1 - \rho(1, 0) - \rho(1, 1))d + (1 - \rho(0, 1) - \rho(1, 1))ed^*] \\ &= \phi[(1 - \rho(1, 0))d + ed^*]. \end{aligned} \quad (52)$$

The buyer's payoff can be evaluated from (44). If $\hat{\pi} < 1$, then reducing d infinitesimally will increase $\hat{\pi}$ in (50), and this increase the buyer's payoff in (44). The buyer would like to attain $\hat{\pi} = 1$ since the seller's participation constraint will still be respected:

$$c(q) \leq \phi[(1 - \rho(1, 0))d + ed^*]. \quad (53)$$

Let the maximum of the buyer's TIOLI value (44) such that the constraints (50), (51) and (53) are respected, in this case be $(U^b)^\dagger$. However, since $\rho(1, 0) \neq 0$, it is easily verified that $(U^b)^\dagger < U_{\{(0,0)\}}^b[\bar{\omega}|\pi(\omega) = \hat{\pi} = 1; \rho(1, 0) = 0] = \sup_{\omega, \rho(1,0)}\{U_{\{(1,0)\}}^b | (50), (51), (53)\} = (U^b)^*$, in which the last equality is attained when $\rho(1, 0) = 0$. This contradicts the claim that $\rho(0, 0) + \rho(1, 0) = 1$ and $\rho(1, 0) \neq 0$ is a component of a PBE.

Case 4. Suppose there are equilibria consisting of $\rho(0, 0) + \rho(0, 1) = 1$ with $\rho(0, 1) \neq 0$, and $\rho(1, 0) = \rho(1, 1) = 0$. The buyer's payoff is such that $U_{\{(0,1)\}}^b = U_{\{(0,0)\}}^b > \max\{U_{\{(1,0)\}}^b, U_{\{(1,1)\}}^b\}$. Given this assumption, we have from (46) that $U_{\{(1,0)\}}^b = U_{\{(1,1)\}}^b$. From (42) and (43), we can derive

$$\hat{\pi} = \frac{\kappa^* - (\phi_{-1}e_{-1}/\phi e - \beta)\phi ed^*}{\beta\sigma\phi ed^*}. \quad (54)$$

From the case that $U_{\{(0,0)\}}^b > U_{\{(1,0)\}}^b$ and (42)-(44), we have:

$$\phi d < \frac{\kappa}{\frac{\phi_{-1}}{\phi} - \beta(1 - \sigma\hat{\pi})}. \quad (55)$$

The buyer's payoff can be evaluated from (43). If $\hat{\pi} < 1$, from (41), we can deduce that the seller's participation constraint is binding. If $\hat{\pi} < 1$, then reducing d^* infinitesimally will increase $\hat{\pi}$ in (54), and this increase the buyer's payoff in (43). The buyer would like to attain $\hat{\pi} = 1$ since the seller's participation constraint will still be respected at that point:

$$c(q) \leq \phi[d + (1 - \rho(0, 1))ed^*]. \quad (56)$$

Let the maximum of the buyer's TIOLI value (43) such that the constraints (54), (55) and

(56) are respected, in this case be $(U^b)^{\dagger\dagger}$. However, since $\rho(1,0) \neq 0$, it is easily verified that $(U^b)^{\dagger\dagger} < U_{\{(0,0)\}}^b[\bar{\omega}|\pi(\omega) = \hat{\pi} = 1; \rho(0,1) = 0] = \sup_{\omega} \{U_{\{(0,1)\}}^b | (54), (55), (56)\} = (U^b)^*$, in which the last equality is attained when $\rho(0,1) = 0$. This contradicts the claim that $\rho(0,0) + \rho(0,1) = 1$ and $\rho(0,1) \neq 0$ is a component of a PBE.

Case 5. Suppose a candidate equilibrium is such that $\sum_{z \in 2^{\{0,1\}^2}} \rho(z) = 1$, $\rho(z) \neq 0$ for all $\{z\} \in 2^{\{0,1\}^2}$, and that $U_{\{(0,1)\}}^b = U_{\{(0,0)\}}^b = U_{\{(1,0)\}}^b = U_{\{(1,1)\}}^b$. Then from (43) and (44), we can derive

$$\hat{\pi} = \frac{\kappa^* - (\phi_{-1}e_{-1}/\phi e - \beta)\phi e d^*}{\beta\sigma\phi e d^*} = \frac{\kappa - (\phi_{-1}/\phi - \beta)\phi d}{\beta\sigma\phi d}. \quad (57)$$

If the payment offered (d, d^*) are such that $\hat{\pi} < 1$, then from the seller's decision rule (41) we can deduce $\omega \equiv (q, d, d^*)$ must be such that the seller's participation/incentive constraint binds:

$$c(q) = \phi[(1 - \rho(1,0) - \rho(1,1))d + (1 - \rho(0,1) - \rho(1,1))ed^*]. \quad (58)$$

However, the buyer can increase his expected payoff in (45) by reducing both (d, d^*) , thus raising $\hat{\pi}$ in (57) while still ensuring that the seller participates, until $\hat{\pi} = 1$, where

$$c(q) \leq \phi[(1 - \rho(1,0) - \rho(1,1))d + (1 - \rho(0,1) - \rho(1,1))ed^*]. \quad (59)$$

Let the maximum of the buyer's TIOLI value (45) such that the constraints (57) and (59) are respected, in this case be $(U^b)^{\ddagger}$. However, since $\rho(1,0), \rho(0,1), \rho(1,1) \neq 0$, it is easily verified that $(U^b)^{\ddagger} < U_{\{(1,1)\}}^b[\bar{\omega}|\pi(\omega) = \hat{\pi} = 1; \rho(0,0) = 1] = \sup_{\omega} \{U_{\{(1,1)\}}^b | (57), (56)\} = (U^b)^*$, in which the last equality is attained when $\rho(0,0) = 1$. This contradicts the claim that $\sum_{z \in 2^{\{0,1\}^2}} \rho(z) = 1$, $\rho(z) \neq 0$ for all $\{z\} \in 2^{\{0,1\}^2}$, is a component of a PBE.

Summary. From Cases 1 to 5, we have shown that the only mixed-strategy Nash equilibrium in the subgame following an offer ω must be one such that $\langle \rho(0,0), \pi \rangle = \langle 1, 1 \rangle$, and that the offer ω satisfies the program in (14) in Proposition 5.

Finally, since $u(\cdot)$ and $-c(\cdot)$ are strictly quasiconcave functions and the inequality constraints in program (14) define a convex feasible set, the program (14) has a unique solution (Sundaram, 2005, Theorem 8.12).

A.3 Proof of Proposition 7

We must consider different cases depending when the endogenous liquidity constraint associated with the local and foreign currency bind or not, given domestic (foreign) inflation rate, money supply, counterfeiting costs and the matching probability. We focus on equilibria that satisfy $\zeta > 0$, $\nu = \nu^* = 0$ so that buyers and sellers trade in DM, and $\mu, \mu^* > 0$. Recall from (14), we denoted

$$\frac{\kappa}{\phi_{-1}/\phi - \beta(1 - \sigma)} =: \bar{\kappa}(\phi_{-1}/\phi); \quad \text{and,} \quad \frac{\kappa^*}{\phi_{-1}e_{-1}/\phi e - \beta(1 - \sigma)} =: \bar{\kappa}^*(\phi_{-1}e_{-1}/\phi e).$$

Hereinafter, let $\bar{\kappa} \equiv \bar{\kappa}(\phi_{-1}/\phi)$, and $\bar{\kappa}^* \equiv \bar{\kappa}^*(\phi_{-1}e_{-1}/\phi e)$ for notational ease.

Case 1(a): $\lambda = 0, \lambda^* = 0$ and $\Pi - \Pi^* > 0$ This case is trivial to check. When both liquidity constraints are not binding, and the foreign currency dominates in rate of return, buyers demand only the foreign currency.

Case 1(b): $\lambda = 0, \lambda^* > 0$ and $\Pi - \Pi^* > 0$ It is easy to show that the resulting system of steady state monetary equilibrium conditions has a unique solution for $\{\zeta, \lambda^*, \phi, q, e\}$. In steady state equilibrium, Π and Π^* are given as the respective (exogenous) money supply growth factors in the home and foreign economies, so that:

$$\lambda^* = \Pi - \Pi^* > 0.$$

From the result in Lemma 9, we have in any monetary equilibrium, a buyer at the end of every CM will make offers of payments (d, d^*) up to the respective limits of their portfolio components (m, m^*) —i.e. $d = m \geq 0$ and $d^* = m^* \geq 0$ —which implies that the multipliers on payment upper-bounds are strictly positive:

$$\mu = \Pi - \beta > 0,$$

$$\mu^* = \Pi^* - \beta > 0.$$

Together, these three conditions also satisfy the equilibrium restriction on interest rates:

$$\mu = \lambda^* + \mu^*.$$

Since we have an equilibrium where monetary exchanges exist in the DM, then the sellers' participation constraint must bind, so that

$$\zeta = \beta\sigma + \Pi - \beta > 0.$$

Also, in this equilibrium the DM quantity of good q exchanged satisfies

$$\sigma\beta \frac{u'(q) - c'(q)}{c'(q)} = \mu = \Pi - \beta > 0. \tag{60}$$

Next, we want to prove that there is coexistence of the home currency with the foreign currency, in spite of the former being dominated in its return, $\Pi > \Pi^*$. By construction the highest sustainable allocation of q is a $q_{FB} > 0$ satisfying the first best trade-off: $u'(q_{FB}) = c'(q_{FB})$. Comparing the first-best condition with the monetary equilibrium condition for q in (60) above, we can easily deduce that $q < q_{FB}$. Since the liquidity constraint on the foreign currency payment is binding, then, from the seller's participation constraint we can re-write as:

$$\phi m = c(q) - \bar{\kappa}^* \geq 0.$$

Suppose to the contrary that the demand for home currency were zero, $m = 0$. However, in this case we have $q = c^{-1}(\bar{\kappa}^*) < q_{FB}$. Since each buyer can increase his lifetime payoff by accumulating more domestic money ($\lambda = 0$) and offering it to the seller in the DM to consume

more q ; and the seller would willingly accept it by producing more q while ensuring that her participation constraint is still binding, then we have in this equilibrium positive demand for home real currency, $\phi m = c(q) - \bar{\kappa}^* > 0$. For a small open economy, we then have $m = M > 0$ in equilibrium.

Finally, since by assumption $\lambda^* > 0$, then the endogenous liquidity constraint on offering/holding $d^* = m^* = M^*$ is binding. Given the supply of domestic money M and foreign money (to domestic buyers) M^* , there is a unique equilibrium nominal exchange rate pinned down by this binding liquidity constraint. Combining this with the seller participation constraint, we have the equilibrium determination of the nominal exchange rate as:

$$e = \frac{M}{M^*} \frac{\bar{\kappa}^*}{c(q) - \bar{\kappa}^*}.$$

Case 1(c): $\lambda > 0, \lambda^* > 0$ and $\Pi - \Pi^* > 0$ It is easy to show that the resulting system of equations has a unique solution for $\{\mu, \mu^*, \phi, e, q, \lambda, \zeta\}$. For a given domestic and foreign inflation rates, money supplies, counterfeited costs and matching probability, the relevant block of the steady state equilibrium conditions is given as follows. First, as in the previous cases,

$$\mu = \Pi - \beta > 0$$

$$\mu^* = \Pi^* - \beta > 0.$$

Second, since the DM-buyer is liquidity constrained in both currencies, then in real terms, he would demand and offer payments up to the limits of both constraints: $\phi M = \bar{\kappa} > 0$ and $\phi e M^* = \bar{\kappa}^* > 0$ as measured in units of the home CM good. From the home currency liquidity constraint, we can solve for

$$\phi = \frac{\bar{\kappa}}{M};$$

and then using this in the foreign currency liquidity constraint, we can derive a unique equilibrium nominal exchange rate

$$e = \frac{M}{M^*} \frac{\bar{\kappa}^*}{\bar{\kappa}}.$$

Finally, the other relevant equilibrium conditions:

$$c(q) = \bar{\kappa}^* + \bar{\kappa};$$

$$\sigma \beta \frac{u'(q) - c'(q)}{c'(q)} = \mu + \lambda;$$

$$\zeta = \beta \sigma + \Pi - \beta + \lambda;$$

$$\lambda^* = \Pi - \Pi^* + \lambda,$$

pin down a unique q, λ, ζ and λ^* , respectively.

Therefore, in this case, there is a determinate monetary equilibrium, with a unique nominal exchange rate, and coexistence of both currencies.

Case 2: $\lambda > 0, \lambda^* = 0$ and $\Pi - \Pi^* < 0$ This case is the symmetric opposite to the previous case. Therefore there can exist a unique steady state e and coexistence of the two currencies, in spite of $\Pi < \Pi^*$.

Case 3(a): $\lambda = \lambda^* = 0$ and $\Pi - \Pi^* = 0$. This case corresponds to the indeterminacy result in Kareken and Wallace. Since both liquidity constraints are not binding, and both currencies yield equal rates of return, then buyers are indifferent as to which currency to hold and sellers' participation constraint binds for any composition of payments offered.

Case 3(b): $\lambda \neq \lambda^* > 0$ and $\Pi - \Pi^* = 0$. In this case, when both liquidity constraints bind, the analysis is similar to Case 1(c) above. Therefore we have coexistence of the two currencies and determinacy of the equilibrium nominal exchange rate.

A.4 Proof of Proposition 8

It suffices to construct a counterexample. Consider Case 3 of Proposition Proposition 7. The relevant block characterizing steady-state monetary equilibrium is

$$\begin{aligned}\mu &= \Pi - \beta, \\ \mu^* &= \Pi^* - \beta, \\ c(q) &= \phi M + e\phi M^*, \\ \phi M &\leq \frac{\bar{\kappa}}{\beta\sigma}, \\ e\phi M^* &\leq \frac{\bar{\kappa}^*}{\beta\sigma}, \\ \sigma\beta \frac{u'(q) - c'(q)}{c'(q)} &= \mu + \lambda, \\ \zeta &= \beta\sigma + \Pi - \beta + \lambda, \\ \lambda^* &= \Pi - \Pi^* + \lambda.\end{aligned}$$

When $\Pi \rightarrow \beta$ and $\Pi^* \rightarrow \beta$ it implies that $\mu \rightarrow 0$ so that

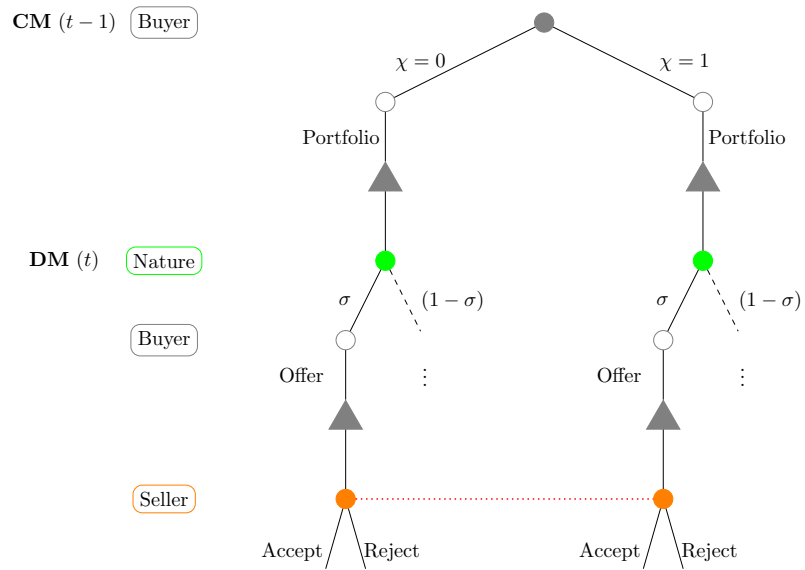
$$\sigma\beta \frac{u'(q) - c'(q)}{c'(q)} = \lambda.$$

Notice that the DM first best q_{FB} , which satisfies $u'(q_{FB}) = c'(q_{FB})$, can only occur if $\lambda = 0$. However, in order for the first best to be a monetary equilibrium, the participation constraint for the seller has to be satisfied and the nominal exchange rate has to be positive. These two conditions are respectively given by

$$\begin{aligned}c(q_{FB}) &\leq \frac{\bar{\kappa}}{\beta\sigma} + \frac{\bar{\kappa}^*}{\beta\sigma}, \\ c(q_{FB}) &\geq \frac{\bar{\kappa}}{\beta\sigma}.\end{aligned}$$

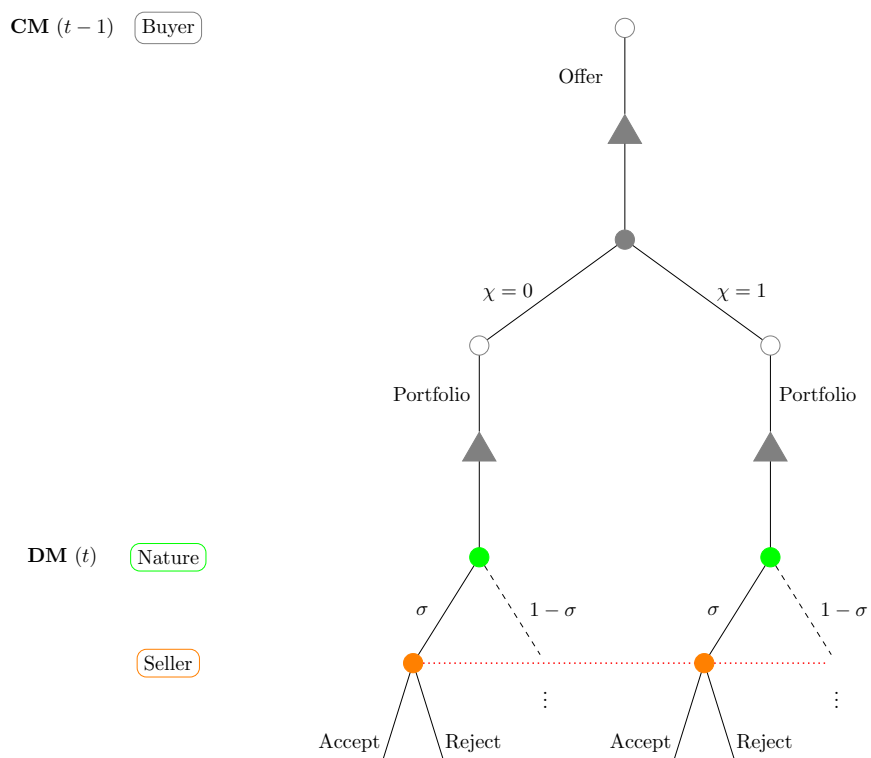
Thus, even when both the domestic and foreign inflation rates converge to the Friedman rule, the DM first best may not be attainable and the nominal exchange rate may not be determinate.

Figure 1: Original extensive-form game.



Note: ● Buyer's discrete decision node; ○ Buyer's continuation to next decision node; ▲ Buyer's continuous decision node; ● Nature's discrete decision node; ● Seller's discrete decision node; ⋯ Information set.

Figure 2: Reverse-order extensive-form game.



Note: • Buyer's discrete decision node; ○ Buyer's continuation to next decision node; ▲ Buyer's continuous decision node; ● Nature's discrete decision node; ● Seller's discrete decision node; ⋯ Information set.