EMPIRICAL RELEVANCE OF AMBIGUITY IN FIRST PRICE AUCTION MODELS

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EMPIRICAL RELEVANCE OF AMBIGUITY IN FIRST PRICE AUCTION MODELS[*]

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ABSTRACT. We study the identification and estimation of first-price auction models with independent private values where bidders are risk averse and there is ambiguity about the valuation distribution. When bidders' preferences are represented by the maxmin expected utility of [Gilboa and Schmeidler, 1989], we provide sufficient conditions for nonparametric identification of the valuation distribution and bidders' attitude toward ambiguity, separately from the risk aversion (CRRA, CARA). We propose a semi-parametric method and apply it to two datasets, one from experimental auctions and the other from USFS timber auctions. We find, for both cases, that bidders are not only risk averse but also ambiguity averse. In addition, we consider the multiplier preferences of [Hansen and Sargent, 2001] and identify the valuation distribution using the same conditions, and show that normalizing, additionally, (any) one quantile of the value, e.g. upper bound of the support, is sufficient to identify the ambiguity parameter separately from the nonparametric utility.

Keywords: first-price auction, identification, Bayesian econometrics, ambiguity aversion.
JEL classification: C11, C44, D44, E61

1. INTRODUCTION

In this paper we study the identification and estimation of first-price auction models with independent private values when risk averse bidders are ambiguous (uncertain) about the valuation distribution. In particular, we consider an environment where bidders consider many distributions as reasonable candidates for the true distribution instead of assuming that they know the correct distribution. One of the main contributions of this paper is to provide sufficient conditions to nonparametrically identify the valuation distribution and bidders' attitude toward ambiguity separately from their attitude toward risk. In this respect we depart from the current literature on empirical auctions by relaxing the (unverifiable) assumption that all bidders commonly know the true valuation distribution and they maximize their expected utility (EU, henceforth).

This common knowledge assumption can be untenable when bidders' appraisal process is complex (e.g. seismic prospecting for mineral/wildcat auctions, and timber auctions)

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or when bidders have not yet learned the market fundamentals after any disturbance (e.g. T-bill auctions after the 2007 financial crisis) leading to ambiguity.[1] For such environments, a model that allows for ambiguity may be preferable to the conventional EU framework not only because it gives a robust inference by nesting EU but also because the presence of ambiguity affects the optimal auction design: (a) the revenue equivalence theorem fails [Lo, 1998]; (b) the first price auction is suboptimal [Bose, Ozdenoren, and Pape, 2006]; and (c) the optimal reserve price decreases with ambiguity [Bodoh-Creed, 2012]. It is, therefore, important to determine the empirical relevance of ambiguity. By proposing a way to do that, we contribute to the empirical auction literature.

Ambiguity in probability judgements has been central in economics since [Keynes, 1921; Knight, 1921], culminating to a position of eminence with [Ellsberg, 1961]. It arises when a decision maker is either unable to pin down the *unique* probability of payoff-relevant-states or is concerned with model misspecification. For example, when model primitives are partially identified, a policy design such as assigning an individual to a treatment or choosing a reserve price in auctions, can be viewed as a decision under ambiguity [Manski, 2000; Aryal and Kim, 2013]. To model ambiguity, we use the maxmin expected utility (MEU, henceforth) of [Gilboa and Schmeidler, 1989]. Under MEU, every bidder has a unique convex and closed set of equally reasonable valuation distributions, and maximizes her expected utility, where the expectation is with respect to the most pessimistic distribution in the set.

One difficulty with the MEU theory is that since it only guarantees the existence of a unique set of distributions that is closed and convex, the same preferences can be represented by many sets of distributions, as shown by [Siniscalchi, 2006]. Hence, inference would be sensitive to how we restrict the shape of the set.[2] We therefore only assume that the set has absolutely continuous distributions around the true distribution and that a bidder with the highest value (on the upper boundary of the value) knows that no rival has a higher value, i.e., no ambiguity at the top.

To formulate bidders' strategy, the set of distributions is common knowledge among the bidders, and so is the fact that values are all independent. The most pessimistic distribution is then also common knowledge, which guarantees the existence of a unique equilibrium with a symmetric and monotonic bidding strategy. We innovate a mapping, the D-function, that bridges the true distribution and the most pessimistic one, which plays an important role in identification and give a testable restriction on ambiguity, i.e., the presence of ambiguity implies the D-function strictly below the 45° degree line, but under EU they coincide.

[1] For instance, wildcat auctions are for offshore areas that haven't been explored, discerning the entire distribution based on seismic prospecting seems unlikely. Likewise, in timber auctions, evaluating the quality of timbers in a tract is a difficult and often error ridden process; see [Athey and Levin, 2001].

[2] Example 1 elaborates more for more on this issue. Pedantically speaking, it is surprising that how we model the set of distributions affects the preferences, once we recognize that under Savage's sure-thing principle, state-independence of preferences is implicitly assumed, which then separates the EU preference from the subjective probability beliefs. That assumption is no longer valid under ambiguity.

(analogous to the Lorenz curve.) Exploiting the bidding strategy, we trace back from the data to the model structures – the valuation distribution, the utility function, and the D-function.

We first show that the nonparametric utility function is not identified even when bidders participation is exogenous (exclusion restriction). However, the valuation distribution and the ambiguity function are nonparametrically identified under the exogenous participation for both constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA) specifications. Since the highest bidder has no ambiguity (no ambiguity at the top), if she overbids, it must be attributable to risk aversion. Hence, the difference of her bidding across two auctions identifies the risk aversion coefficient. Then, by comparing bid quantiles across two auctions, we identify the D-function and the valuation distribution. The exclusion restriction is used by [Guerre, Perrigne, and Vuong, 2009] to nonparametrically identify the utility function under EU.

As identification relies on the highest valued bidder our estimator will be based on the support of the observed data that in turn depends on some other parameters of interest, leading to what is known as an nonregular model. So following the recommendation of [Hirano and Porter, 2003], who show that in such cases MLE is generally inefficient but Bayes estimator is efficient according to the local asymptotic minmiax criteria for conventional loss functions, we use semi-parametric Bayes estimator. Our estimator is based on Dirichelt process of [Ferguson, 1973] and the random Bernstein polynomial of [Petrone, 1999a,b].

The estimation of a auctions with ambiguity is new in the literature, so before we analyze the field data, we implement our estimation method in a sample from experimental auctions of [Dyer, Kagel, and Levin, 1989] where values are drawn from a uniform distribution. An advantage of using this data is that the experiment is simple and has also been used by [Bajari and Hortaçsu, 2005] to estimate the risk aversion (CRRA) coefficient. We expect our estimated of valuation distribution to be uniform and the risk aversion parameter to confirm with the estimates by [Bajari and Hortaçsu, 2005]. We not only match both these parameters but we also find that the estimated ambiguity-function is significantly below the identity mapping ($45°$ - line), i.e. there is strong evidence of ambiguity.

Next we study the Timber auctions conducted by the U.S. Forest service to test for presence of ambiguity. As mentioned earlier, even though it has been acknowledged that timber auction has a relatively complex appraisal process [Athey and Levin, 2001], most of the empirical analysis ignores the possibility that the bidders might not know the correct distribution. Furthermore, another feature of the data that is important from the point of ambiguity is the low reserve price, which has been largely ignored by earlier works. For instance [Lu and Perrigne, 2008] argue that these reserve prices do not bind and can be ignored. [Aryal and Kim, 2013], however, show that when the seller is ambiguity averse, it is decision theoretically optimal to choose a low reserve price. So the argument goes, if

USFS can be ambiguous about the distribution and if the appraisal process is complex, it is highly plausible that bidders are also ambiguous. Hence, the timber auction provide a natural setting to implement our model for ambiguity averse bidders. The posterior distribution of the risk aversion parameters are consistent with the point estimates in [Lu and Perrigne, 2008] and the estimates support the presence of ambiguity among bidders.

We show how the identification result can be extended to auctions with (non separable) unobserved heterogeneity and MEU preferences as long as there are three bidders in each auction. The identification strategy follows that of [Hu, McAdams, and Shum, 2011]. Lastly, we consider identification of the Multiplier Preference (MP) model of [Hansen and Sargent, 2001] as an competing model of ambiguity. Since MP can be transformed into a risk averse model without ambiguity [Strzalecki, 2011], the nonparametric identification of the utility function follows from [Guerre, Perrigne, and Vuong, 2009], and to identify the ambiguity aversion parameter we show it is sufficient to normalize any one quantile of valuation, such as the upper bound.[3]

Therefore our paper is related and contributes to the literature on empirical auction models studied by [Paarsch, 1992; Guerre, Perrigne, and Vuong, 2000, 2009; Athey and Haile, 2002; Haile and Tamer, 2003], among others. Our paper is also related to papers that use experiments to elicit ambiguity preferences in decision theory [Halevy, 2007; Ahn, Choi, Gale, and Kariv, 2011] and in auction [Chen, Katuščák, and Ozdenoren, 2007] and on measuring expectation (or beliefs) [Manski, 2004]. We, therefore aim to contribute to the literature that recognizes the importance of ambiguity/robustness in economic modeling as summarized by [Hansen and Sargent, 2011].

In the next section we describe the model with MEU and the identification of the model structure, in section 3 we present estimation results using experimental Timer auction data. In section 4 we consider unobserved heterogeneity; identification of multiplier preferences is presented in section 6 and we conclude. All technical details are collected in the Appendices.

2. Model And Identification

An indivisible private valued object is to be allocated to one of $n \geq 2$ bidders without reserve price. Each bidder i observes only her own value v_i and bids b_i. The highest bidder wins the object and gets utility $u(v_i - b_i)$ while the rest get $u(0) = 0$, where $u(\cdot)$ is an increasing and strictly concave utility function. The objective of bidder i with value v_i is to solve:

$$\max_{b_i} \{u(v_i - b_i) \times Pr(win)\} \equiv \max_{b_i} \left\{u(v_i - b_i) \times Pr(b_i \geq b_j, j \neq i)\right\}. \tag{1}$$

[3] Although MP model is widely used in macroeconomics, as far as we know, that literature has not considered identification of this ambiguity parameter. Without identification a researcher has to select an arbitrary value or use some other rule such as the detection probability error model of [Hansen and Sargent, 2010]. The choice of this parameter is crucial, as the planner's policy function varies with it; see [Svensson, 2001; Giordani and Söderlind, 2004].

The nature draws values *i.i.d.* from $F_0(\cdot|n, W)$ defined over $[\underline{v}(W,n), \overline{v}(W,n)]$ but the bidders *do not* know this distribution. Here, $W \in \mathcal{W} \subset \mathbb{R}^L$ is a vector of auction covariates that is observed by both the bidders and the researcher. For notational ease, we shall suppress the dependence on W until Section 3.3. Since, bidders do not know $F_0(\cdot|n)$ they cannot compute the "winning probability," that is essential to solve (1). Following the literature on decision under ambiguity, we assume that even though the bidders do not know $F_0(\cdot|n)$ they consider many equally reasonable distributions.

Let $\mathcal{P}_n$ be a *convex* set of all strictly increasing continuous distribution functions defined over $[\underline{v}(n), \overline{v}(n)]$ for a given $n \in \mathcal{N} := \{n \in \mathbb{N} : 2 \leq n < \infty\}$ such that $F_0(\cdot|\cdot) \in \mathcal{P}_n$.[4] There is also a need to model each bidder's beliefs about others' beliefs about the set of distributions because it will affect the equilibrium behavior. First, we assume that the number of bidders n in an auction is common knowledge. Second, we also assume that even though the bidders do not know$F_0(\cdot|n)$ it is commonly known that the values are all drawn from only one distribution from the set $\mathcal{P}_n$. This assumption of symmetry-in-beleifs among the bidders keeps the model tractable because we do not have to model higher order beliefs explicitly. A justification for this is that bidders have access to a common training data from which they can learn, e.g. in timber auctions bidders can "cruise" the same tracts before bidding. Collectively, we make the following assumptions:

Assumption 1. *It is common knowledge among the bidders that:*

(1) *There are $n \in \mathcal{N}$ bidders with an identical utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ with $u' > 0$, $u'' < 0$, and $u(0) = 0$.*
(2) *Their values $v_1, \ldots, v_n$ are independently and identically distributed.*
(3) *The true valuation distribution $F_0(\cdot|n) \in \mathcal{P}_n$ with density $f_0(\cdot|n) > 0$ is unknown to the bidders, but any information about $F_0(\cdot|n)$ other than realized values is shared among the bidders.*

We focus only on a symmetric pure strategy Bayesian Nash equilibrium. In particular, every bidder conjectures that her opponents use a strictly increasing (pure) bidding strategy, and announces a bid that is a best response to that conjecture and at equilibrium the conjecture turns out to be true.[5] When the distribution $F_0(\cdot|n)$ is common knowledge, there is a unique symmetric Bayesian Nash equilibrium in pure strategy characterized by a strictly increasing bidding function $\beta_n : [\underline{v}(n), \overline{v}(n)] \to \mathbb{R}_+$; see Theorem 6 [Athey, 2001]. This bidding strategy maps the latent value to the observed bid. [Guerre, Perrigne, and Vuong, 2000] shows that when bidders are risk neutral, this map can be inverted to link each bid to a unique value, thereby identifying $F_0(\cdot|n)$. [Guerre, Perrigne, and Vuong, 2009; Campo, Guerre, Perrigne, and Vuong, 2011] extend this result to allow for risk averse

[4] The convexity assumption is without loss of generality because all preferences that are valid with a given set of distributions are also valid for the convex hull of this set. In other words, the partial order of preference is invariant to any convex combination of members in the set.

[5] In our model, bidders do not have the option of using "ambiguous strategies", i.e. they do not have access to a subjective randomizing device (such as Ellsberg urns); see [Bade, 2011; Riedel and Sass, 2011].

bidders. In the remaining of this section, we extend these results to the MEU and MP representations.

Maxmin Expected Utility. The seminal article [Gilboa and Schmeidler, 1989] proposes an axiomatic representation of preferences for decision makers (bidders) with multiple priors (valuation distribution) about the state of nature (opponents' values). We assume

Assumption 2. *The preference ordering of each bidder satisfy*

(1) *Assumptions A1-A6 in [Gilboa and Schmeidler, 1989].*
(2) *Monotone Continuity.*

Assumption 2 (1) coincides with axioms in the expected utility (EU) , except it allows decision makers to weakly prefer any convex combination of indifferent lotteries to each individual one instead of restricting the combination to be indifferent– *ambiguity aversion*, and uses a weaker version of independence. Let Ω be the set of the states of nature, $\tilde{u}(\cdot)$ the utility function, and $\mathcal{A}$ the set of all feasible actions. [Gilboa and Schmeidler, 1989] shows that a decision maker's preference ordering satisfies assumption 2 if and only if there is a unique set of distributions $\mathcal{C}$ over Ω such that she prefers an action a to b whenever $\min_{P\in\mathcal{C}} \mathbb{E}_P\tilde{u}(a) \geq \min_{P\in\mathcal{C}} \mathbb{E}_P\tilde{u}(b)$. We begin by proposing a way to adapt the set of distributions to represent the strategic effects of ambiguity.

If every bidder had a different set of distributions, we would have to explicitly model the beliefs of other bidders about that set, and that bidder's beliefs about what others believe. As mentioned earlier, to gain traction we follow the tradition of [Harsanyi, 1967], and interpret the auction as a game of imperfect information among bidders, where although all bidders are ambiguous about the true distribution, it is common knowledge that there is one unique set of distribution that contains the true distribution. From assumptions 1 and 2, this implies that every bidder uses the most pessimistic distribution to determine her expected utility and chooses a bid accordingly.

Then the next step is to model a set of distributions that is sufficiently general enough to model ambiguity but at the same time should always has a unique lowest/worst distribution that first order stochastically dominates all other distributions in the set; see [Gilboa and Marinacci, 2010]. Before we define such a set we begin with an example that demonstrates how the set of distribution we choose can affect our inference. To that end, we consider the ε-contaminated model that is widely used, in economics and in robust statistics, to model the set of distributions.[Nishimura and Ozaki, 2006; Kopylov, 2008] provide an axiomatic justification for using ε-contamination set to model ambiguity in decision making.[6] Then we show that this model is observationally equivalent to a first price auction without ambiguity.

[6] This ε-contaminated model is widely used in the literature; see [Huber, 1973; Berger, 1985; Berger and Berliner, 1986; Nishimura and Ozaki, 2004; Bose, Ozdenoren, and Pape, 2006; Bose and Daripa, 2009; Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio, 2011; Aryal and Stauber, 2013].

Example 1. *Let $\varepsilon \in (0,1)$ be given and is commonly known to the bidders who, without loss of generality, are assumed to be risk neutral. For a fixed n, under the ε-contaminated model, the set of distribution is defined as*

$$\Gamma' := \{F(\cdot|n) : F(\cdot|n) = (1-\varepsilon)F_0(\cdot|n) + \varepsilon R(\cdot|n) \text{ with } R(\cdot|n) \in \mathcal{P}_n\},$$

where $R << F_0$ is a distribution. Let $F^*(\cdot|n)$ be the most pessimistic distribution and $f^*(\cdot|n)$ its density. Then we know

$$F^*(v|n) = (1-\varepsilon)F_0(v|n) + \min_{R\in\Gamma'} R(v|n) = \begin{cases} (1-\varepsilon)F_0(v|n), & v < \overline{v} \\ 1, & v = \overline{v}. \end{cases}$$

so that

$$\arg \max_{x\in[\underline{v}(n),\overline{v}(n)]} [v - \beta_n(x)] \left[(1-\varepsilon)F_0(x|n)\right]^{n-1} = \arg \max_{x\in[\underline{v}(n),\overline{v}(n)]} [v - \beta_n(x)] F_0(x|n)^{n-1},$$

which means the solution to MEU model with the Γ' as the set of distribution is also the solution to EU model.

Intuitively, this transpires because ambiguity, as measured by ε, scales the true distribution for all bidders by a (multiplicative) factor of $(1-\varepsilon)$, and hence does not affect the relative probability of winning. This means the the inverse hazard function that governs the bids is independent ε and hence bidders can ignore ε. Therefore, the model is with ambiguity and is observationally equivalent to the one without ambiguity, and the identification of $F_0(\cdot|n)$ follows from [Guerre, Perrigne, and Vuong, 2000] and is formally stated below:

Lemma 1. *If the set of distributions is given by ε-contamination of $F_0(\cdot|n)$ then the model is observationally equivalent to the first price auction model without ambiguity.*

Therefore we want the set of distribution to be more general than Γ' but not contain any degenerate (Dirac) distribution.[7] So we restrict ourselves to distributions that are absolutely continuous with respect to $F_0(\cdot|n)$. We begin our model by formalizing these assumptions (symmetry, non-multiplicative ambiguity and absolute continuity).

Let $\mathcal{Q}_n \subseteq \mathcal{P}_n$ be a convex subset of all the probability distributions such that (a) $\mathcal{Q}_n \ni F_0(\cdot|n)$ and (b) every element $F(\cdot|n) \in \mathcal{Q}_n$ is strictly increasing and differentiable. Hence, $\mathcal{Q}_n$ is convex and weak* compact [Parthasarathy, 1967] and contains distributions that are absolutely continuous with respect to $F_0(\cdot|n)$; see Theorem 6 [Gilboa and Marinacci, 2010]. In fact the assumption 2 implies the set $\mathcal{Q}_n$ is weakly compact [Chateauneuf, Maccheroni, Marinacci, and Tallon, 2005].

Since the probability of a bidder winning at a bid b depends on the joint likelihood of everybody else bidding less than b, which in turn depends the joint distribution of the values, we have to define such a distribution. In an auction with n bidders, for every bidder

[7] If degenerate distributions are allowed then the most pessimistic distribution will be the one that put all the mass on the $\overline{v}$ so that bidding zero will be optimal.

i, define

$$\tilde{\Gamma}_{-i} = co\{\underbrace{F(\cdot|n) \times F(\cdot|n) \times \ldots F(\cdot|n)}_{(n-1)times} : F \in \mathcal{Q}_n\},$$

as the set of joint probability distributions of all $(n-1)$ bidders valuation, where coA denotes the convex hull of the set A. The set $\tilde{\Gamma}_{-i}$ represents bidder $i's$ beliefs about other bidders' values. By assumption bidders are symmetric, this set is the same for all bidders, i.e. $\tilde{\Gamma}_{-i} = \tilde{\Gamma}_{-j}$ for all $i \neq j$ we may drop out the index $-i$. Moreover, since every member of $\tilde{\Gamma}$ is a joint distribution of a random sample of size $n-1$, for simplicity, we use Γ to refer to the unique set of marginal distributions that are associated with the set of joint distributions $\tilde{\Gamma}$.[8]

In line with [Gilboa and Schmeidler, 1989], the size of this set determines the degree of ambiguity. For example, if it is singleton, there is no ambiguity, and the model becomes EU. Alternatively, if Γ is a closed ball with a diameter $\varepsilon > 0$ around $F_0(\cdot|n)$ for an appropriately chosen metric, the larger ε, the bigger Γ, which means more ambiguity. We assume that all the bidders are symmetric in terms of the ambiguity and information structure.

Assumption 3. *The fact that $F_0(\cdot|n) \in \Gamma$ and the set Γ itself are common knowledge among all bidders.*

Though every bidder can compute the lowest probability of winning by assumption 3, the econometricians do not know Γ. We are interested in inferring this lowest probability of Γ and therefore the true valuation distribution from the bid, under the assumptions that bidders choose their bids to maximize their expected utility with respect to the worst distribution. Since Γ is convex and weakly compact, its lower envelope $F^*(\cdot|n)$ belongs to Γ and unique, and so is the density $f^*(\cdot|n)$ almost everywhere.[9]

Let $D : [0,1] \to [0,1]$ that solves the min part of the bidder's objective, i.e. $D\left[F_0(v|n)\right] := F^*(v|n) = \min_{F\in\Gamma} F(v|n), \forall v \in [\underline{v}(n), \overline{v}(n)]$. Equivalently, for all $p \in [0,1]$

$$D(p) := F^*\left[F_0^{-1}(p|n)|n\right], \tag{2}$$

and hence it maps the true probability $F_0(\cdot|n)$ to the most pessimistic one $F_0^*(\cdot|n)$ so $D(p) \leq p$. Whenever there is ambiguity, $D(p)$ will be less than p, for all $p \in (0,1)$ so that the distance of $D(\cdot)$ from the 45° line measures the extent of ambiguity.

Now, we study the equilibrium bidding function and the identification of model primitives. We focus on the equilibrium bidding strategy that is a best response to a bidder when all other bidders adopt a strictly increasing, symmetric pure strategy bidding function. [Athey, 2001] shows that the best response itself is a strictly increasing bidding strategy,

[8] Although we remain agnostic about the nature of symmetry and how all bidders have access to the same data, a formal justification and representation might be possible if we follow [Klibanoff, Mukerji, and Seo, 2012], something that is beyond the scope of this paper.

[9] It is a lower envelope in the sense that it is first-order-stochastically-dominated by every distribution in the set, i.e. $F(v|n) \geq F^*(v|n)$ for all $v \in [\underline{v}(n), \overline{v}(n)]$ for all $F(\cdot|n) \in \Gamma$.

$\beta_n(\cdot)$, such that a bidder with a value v behaves as if her value is z that solves

$$\max_{x\in\mathbb{R}_+}\min_{F\in\Gamma} u\left[v-\beta_n(x)\right] F(x|n)^{n-1} = \max_{x\in\mathbb{R}_+} u\left[v-\beta_n(x)\right] D\left[F_0(x|n)\right]^{n-1}. \tag{3}$$

The first-order condition with respect to z gives

$$-u'\left[v-\beta_n(x)\right]\beta_n'(z)D\left[F_0(x|n)\right] + u\left[v-\beta_n(x)\right](n-1)D'\left[F_0(x|n)\right]f_0(x|n) = 0$$

at $x = v$. By rearranging terms,

$$\frac{u\left[v-\beta_n(v)\right]}{u'\left[v-\beta_n(v)\right]} = \frac{D\left[F_0(v|n)\right]}{D'\left[F_0(v|n)\right]}\left[\frac{1}{(n-1)f_0(v|n)/\beta_n'(v)}\right].$$

Let $\lambda(x) := u(x)/u'(x)$ for $x \in \mathbb{R}$, then $\lambda'(\cdot) \geq 1$ and hence it is invertible. Let $H(p) := D(p)/D'(p)$ for $p \in [0,1]$, or alternatively

$$H(p) = F^*\left[F_0^{-1}(p|n)\Big|n\right]\frac{f_0\left[F_0^{-1}(p|n)\Big|n\right]}{f^*\left[F_0^{-1}(p|n)\Big|n\right]}. \tag{4}$$

Note that $H(\cdot)$, unlike $\lambda(\cdot)$, is not necessarily monotone with a slope greater than 1. Substituting $\lambda(\cdot)$ and $H(\cdot)$ in the first order condition gives

$$\lambda\left[v-\beta_n(v)\right] = \frac{H\left[F_0(v|n)\right]}{(n-1)f_0(v|n)/\beta_n'(v)} \tag{5}$$

Identification. Let $G(\cdot|n)$ be the distribution of equilibrium bid $b := \beta_n(v)$ for $v \sim F_0(\cdot|n)$, i.e., $G(b|n) = F_0[\beta^{-1}(b)|n]$ and its density is

$$g(b|n) := \frac{f_0[\beta^{-1}(b)|n]}{\beta_n'[\beta_n^{-1}(b)]}.$$

Let v_γ and b_γ be the γ-th quantile of the value and the equilibrium bid. Since $\gamma = F_0(v_\gamma|n) = G[\beta_n(v_\gamma)|n] = G(b_\gamma|n)$, for every $\gamma \in [0,1]$, (5) gives

$$\lambda(v_\gamma - b_\gamma) = \frac{H(\gamma)}{(n-1)g(b_\gamma|n)} \tag{6}$$

for every $\gamma \in [0,1]$. Under the i.i.d. assumption, $g(\cdot|n)$ is nonparametrically identified from the bid data, but the model primitives are not in general identified without additional assumptions, including the ones on the set Γ.

In the remaining subsection, we explore sufficient, yet plausible, conditions under which the model primitives are identified. The following proposition establishes a negative result that does not depend on the structure of the set Γ.

Proposition 1. *Under assumptions 1, 2 and 3, the valuation distribution $F_0(\cdot|n)$ is not identified by the knowledge of bid distribution, i.e., $g(\cdot|n)$.*

Proof. It would be sufficient just to consider risk neutral bidders. Then, (6) can be written as $(v_\gamma - b_\gamma)(n-1)g(b_\gamma|n) = H(\gamma)$. Since $\gamma = F_0(v_\gamma|n)$,

$$[F_0^{-1}(\gamma|n) - b_\gamma](n-1)g(b_\gamma|n) = H(\gamma), \qquad \forall\gamma \in [0,1].$$

This one equation has two unknowns: $F_0^{-1}(\gamma)$ and $H(\gamma)$. □

In view of this result, we consider auctions with exogenous participation and assume the following.

Assumption 4. *Exogenous Participation:* $n \in \mathcal{N}, F_0(\cdot|n) = F_0(\cdot)$.[10]

Assumption 4 is widely used in the literature by [Athey and Haile, 2002; Guerre, Perrigne, and Vuong, 2009; Aradillas-Lopez, Gandhi, and Quint, 2012] among others and is equivalent to assuming that there is some n' potential bidders with values $(v_1, \ldots, v_{n'})$ out of which a random subset of $n \leq n'$ bidders participate in a given auction. This identifying assumption is appropriate for our purpose because it holds in both the data sets (experiments and timber auction) we use to estimate the model. While in the experimental data the number of bidders are exogenously chosen by the experimenter, for timber auction [Aradillas-López, Gandhi, and Quint, 2011] find the assumption to be valid.

Unfortunately, however, when the utility function is unspecified, this exclusion restriction is not sufficient to identify the model structure as shown below.

Proposition 2. *Suppose the econometrician identifies $g(\cdot|n_1)$ and $g(\cdot|n_2)$ where $n_j \in \mathcal{N}$ with $n_1 \neq n_2$. Then under assumptions 1–4, the model structure $[u(\cdot), F_0(\cdot)]$ is not nonparametrically identified.*

Proof. We begin by stating (without a proof) the rationalizability lemma.

Lemma 2. *Let $\mathbf{G}_j(\cdot|n_j)$ be the joint distribution of $(b_1^j, b_2^j, \ldots, b_{n_j}^j)$, conditional on n_j for $j = 1, 2$. There exists an IPV auction model with risk aversion and maxmin expected utility $[u(\cdot), F_0(\cdot)]$ that rationalizes both $\mathbf{G}_1(\cdot|n_1)$ and $\mathbf{G}_2(\cdot|n_2)$ if and only if the following conditions hold:*

(1) $\mathbf{G}_j(b_1^j, \ldots, b_{n_j}^j|n_j) = \prod_{i=1}^{n_j} G_j(b_i^j|n_j)$, *where $G_j(\cdot|n_j)$ is the bid distribution form auction with n_j bidders.*

(2) $\exists \lambda : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $\exists H : [0,1] \to \mathbb{R}_+$ *such that* $\lambda(0) = 0, H(0) = 0$ *and* $\lambda'(\cdot) \geq 1$ *such that* $\xi'(\cdot) > 0$ *on* $[\underline{b}, \overline{b}]$ *where* $\xi(b, u, G, n, H)$:
   (a) $\xi(b, u, G_j, n_j) = b + \lambda^{-1}\left[\frac{H(G_j(b^j|n_j))}{(n_j-1)g_j(b^j|n_j)}\right], j = 1, 2.$
   (b) $\forall \gamma \in [0,1], b_\gamma^1 + \lambda^{-1}\left[\frac{H(\gamma)}{(n_1-1)g(b_\gamma^1|n_1)}\right] = b_\gamma^2 + \lambda^{-1}\left[\frac{H(\gamma)}{(n_2-1)g(b_\gamma^2|n_2)}\right].$

Following [Guerre, Perrigne, and Vuong, 2009] we can identify $\lambda^{-1}(\cdot)$. Let $[F(\cdot), \lambda(\cdot), H(\gamma) := \gamma]$ and $[\tilde{F}(\cdot), \lambda(\cdot), \tilde{H}(\gamma) := \gamma^\iota]$, with $\iota \in (0,1)$ be two model structures. Let $\tilde{F}(\cdot)$ be the distribution of $\tilde{v}$ defined as follows: For every quantile $\gamma \in (0,1]$ compute $v(\gamma) = F^{-1}(\gamma)$ and determine $b_\gamma^j = \beta[v_\gamma, F(\cdot), n_j, H]$ and

$$\tilde{v}_\gamma = b_\gamma^j + \lambda^{-1}\left[\frac{\gamma^\iota}{(n_j-1)g_j(b_\gamma^j|n_j)}\right].$$

[10] This also means that the set $\mathcal{P}_n$ is the same for all $n \in \mathcal{N}$ and because Γ will also be the same, so will $F^*(\cdot)$.

Then, it is clear that the two model structures satisfy Lemma 2 and hence are observationally equivalent. □

Without ambiguity the model structure $[u(\cdot), F_0(\cdot)]$ is just-identified by the knowledge of $g(\cdot|n_1)$ and $g(\cdot|n_2)$ with $n_1 \neq n_2$, and with ambiguity we have to identify an extra parameter, the ambiguity-function $D(\cdot)$. In view of this result, we restrict ourselves to parametric families of utility functions, CRRA and CARA, that are most widely used in the empirical literature, nesting risk neutral bidders as a special case;

Assumption 5. *(CRRA) The utility function is CRRA, i.e.,*

$$u(w) = \frac{w^{1-\theta}}{1-\theta} \quad \text{with } \theta \in [0,1)$$

Assumption 6. *(CARA) The utility function is CARA, i.e.,*

$$u(w) = \begin{cases} [1-\exp(-w\theta)]/\theta & \text{if } \theta > 0 \\ w & \text{if } \theta = 0 \end{cases}$$

Under assumptions 5 and 6, $\lambda(\cdot)$ becomes

$$\lambda(w) = \begin{cases} \frac{w}{1-\theta}, & \text{under CRRA} \\ \frac{1-\exp(-w\theta)}{\theta\exp(-w\theta)}, & \text{under CARA,} \\ w, & \text{under risk neutrality.} \end{cases}$$

As propositions 1 and 2 argue, the model is not identified without the exclusion restriction. This is true even with the parametrized utility functions. It would be, therefore, useful to understand the source of nonidentification without the exclusion restriction. Consider the CRRA utility, since it is strictly increasing and concave, choosing a bid to maximize MEU is equivalent to choosing a bid to maximize the certainty equivalence corresponding to a worse distribution, i.e. [11]

$$\arg\max_x \frac{[v-\beta_n(v)]^{1-\theta}}{1-\theta} D[F_0(x|n)]^{n-1} = \arg\max_z [v-\beta_n(x)] D[F_0(x|n)]^{\frac{n-1}{1-\theta}},$$

which means a risk averse bidder with CRRA utility with parameter θ would bid the same as a risk neutral bidder who is more pessimistic i.e. $D(\cdot)^{\frac{1}{1-\theta}}$. Hence, as far as the bid data is concerned, there is a substitutability between risk aversion and the "pessimism". This is because both lead to over-bidding in the same way and hence cannot be disentangled by the data. Consider now a bidder with the highest value, $\overline{v}$. She should not distort her winning probability, as she knows that she will win the auction with probability one (irrespective of the true distribution) because the support is common knowledge, and hence there is no ambiguity.

Assumption 7. *No ambiguity at the top:* $D(1) = 1$.

[11] And given our notation of the worst distribution, the certainty equivalence $c.e(v, z, D, F_0, \theta)$ solves $c.e(v, v, D, F_0)^\theta = [v - \beta(z)]^\theta [D(F_0(v|N))]^{n-1}$.

This can be achieved if the set Γ is further restricted to consist of only those distributions F with inverse Mill's ratio that is equal to the true inverse Mill's ratio at $\overline{v}$, i.e. $\frac{f(\overline{v})}{F(\overline{v})} = \frac{f_0(\overline{v})}{F_0(\overline{v})}$. Once the risk aversion coefficient is identified by the highest bidder's bidding behavior, the exclusion restriction is sufficient to identify the bidder's attitude toward ambiguity and also the valuation distribution. We now formally establish the identification of the model primitives, starting with CRRA specification with risk neutral case as its corollary and CARA specification.

Proposition 3. *Suppose that bidders' utility function u is CRRA, i.e, assumption 5. Under assumptions 1, 2, 3, 4 and 7, the valuation distribution $F_0(\cdot)$, the utility function u, and the function D are identified by $g(\cdot|n_1)$ and $g(\cdot|n_2)$ where $n_1, n_2 \in \mathcal{N}$ with $n_1 < n_2$.*

Proof. Under assumption 5, $\lambda(x) = \frac{x}{1-\theta} \Leftrightarrow \lambda^{-1}(y) = (1-\theta)y$. From (6), we get

$$v - b = \lambda^{-1}\left\{\frac{H[G(b|n)]}{(n-1)g(b|n)}\right\} = (1-\theta)\left\{\frac{H[G(b|n)]}{(n-1)g(b|n)}\right\}$$

For each quantile $\gamma \in [0,1]$, let $v_\gamma \in [\underline{v}, \overline{v}]$ such that $F_0(v_\gamma) = \gamma$, and $b^j_\gamma := \beta_{n_j}(v_\gamma)$. Then, since $G(b^j_\gamma|n_j) = G[\beta_{n_j}(v_\gamma)|n_j] = F_0(v_\gamma) = \gamma$, for each $\gamma \in [0,1]$, we have

$$v_\gamma = b^j_\gamma + \frac{(1-\theta)H(\gamma)}{(n_j-1)g(b^j_\gamma|n_j)}. \tag{7}$$

where $j \in \{1,2\}$. Equating the quantiles for v under two auctions, we get

$$(1-\theta)H(\gamma) = (b^2_\gamma - b^1_\gamma)\left[\frac{1}{(n_1-1)g(b^1_\gamma|n_1)} - \frac{1}{(n_2-1)g(b^2_\gamma|n_2)}\right]^{-1},$$

which when evaluated at $\gamma = 1$ identifies θ since $H(1) = 1$. Once θ is identified, identification of $H(\gamma)$ is straightforward, which then identifies

$$D(\gamma) = \exp\left[-\int_\gamma^1 \frac{1}{H(t)}dt\right],$$

as well as $F_0(\cdot)$ from equation (7). Moreover, $F^*(v) = D[F_0(v)]$. □

As mentioned above, the highest bidder's bidding behavior identifies θ_{crra}. After controlling the effect of risk aversion, any deviation from the standard model explains bidders' attitude toward ambiguity, identifying D, from which the identification of F_0 follows. An immediate corollary is the identification with risk neutral bidders, which is the case of $\theta = 0$.

Corollary 2. *Suppose that bidders are risk neutral. Under assumptions 1, 2, 3 and 4, the valuation distribution $F_0(\cdot)$ is identified by the knowledge of bid distributions $g(\cdot|n_1)$ and $g(\cdot|n_2)$ where $n_1, n_2 \in \mathcal{N}$ with $n_1 < n_2$.*

We now establish the identification of the auction model under CARA.

Proposition 4. *Suppose that bidders' utility function u is CARA, i.e, assumption 6. Under assumptions 1, 2, 3, 4, and 7, the valuation distribution $F_0(\cdot)$, the utility function u, and the function $D(\cdot)$ are identified by $g(\cdot|n_1)$ and $g(\cdot|n_2)$ where $n_1, n_2 \in \mathcal{N}$ with $n_1 < n_2$.*

Proof. As in the previous proof, for each quantile $\gamma \in [0,1]$, let $v_\gamma \in [\underline{v}, \overline{v}]$ such that $F_0(v_\gamma) = \gamma$, and $b^j_\gamma := \beta_{n_j}(v_\gamma)$. Since

$$\lambda(w) = \frac{1-\exp(-w\theta)}{\theta\exp(-w\theta)} = \frac{1}{\theta}[\exp(w\theta)-1],$$

its inverse is

$$\lambda^{-1}(y) = \log(1+\theta y)/\theta.$$

From (6), for every $\gamma \in (0,1]$, and $j \in \{1,2\}$ we have

$$v_\gamma = b^j_\gamma + \frac{1}{\theta}\log\left[1 + \frac{H(\gamma)\theta}{(n_j-1)g(b^j_\gamma|n_j)}\right]. \tag{8}$$

Since v_γ is the same for both $j = 1,2$, we can equate the two equations to get

$$(b^2_\gamma - b^1_\gamma)\theta = \log\left[1 + \frac{H(\gamma)\theta}{(n_1-1)g(b^1_\gamma|n_1)}\right] - \log\left[1 + \frac{H(\gamma)\theta}{(n_2-1)g(b^2_\gamma|n_2)}\right] \tag{9}$$

Clearly, $\theta = 0$, i.e. linear utility, solves this equation. For identification we want to show that there is another $\theta \neq 0$ that also solves the equation. The left hand side of (9) as a function of θ is linear in θ, starts at the origin and is strictly increasing with a constant slope of $(b^2_1 - b^1_1)$. Let $m(\theta)$ be the right hand side of (9) and $R_j := [(n_j-1)g(b^j_1|n_j)]^{-1}$, then because $H(1) = 1$ (from assumption 7) $m'(\theta) = [R_1 - R_2]/[(1+\theta R_2)(1+\theta R_1)] > 0, \lim_{\theta\to 0} m(\theta) = 0, \lim_{\theta\to\infty} m(\theta) = \log\left(\frac{R_1}{R_2}\right) < \infty$, and $m''(\theta) < 0$. That is, $m(\theta)$ is strictly increasing and strictly concave, that starts at the origin and converges to a finite constant from below. Thus, if $m'(0)$ is greater than the slope of the LHS of (9), which is $(b^2_1 - b^1_1)$ then there is a unique $\theta > 0$ that solves (9). From Lemma 2 condition 2, we know $\lambda'(\cdot) \geq 1$ and so $(\lambda^{-1})'(\cdot) \in (0,1)$. Then aggressive bidding in auction with n_2 bidders imply the rent under n_1 auction is greater than under n_2 auction and hence $v_1 - b^1_1 > v_1 - b^2_1 \Leftrightarrow \lambda^{-1}(R_1) > \lambda^{-1}(R_2) \Leftrightarrow R_1 > R_2$. Therefore,

$$b^2_1 - b^1_1 = \lambda^{-1}(R_1) - \lambda^{-1}(R_2) < R_1 - R_2 = m'(0),$$

as desired. Once θ is identified, we can identify $H(\gamma)$ from (9) as

$$H(\gamma) = \frac{\exp((b^2_\gamma - b^1_\gamma)\theta) - 1}{\theta\left[\frac{1}{(n_1-1)g(b^1_\gamma|n_1)} - \frac{\exp((b^2_\gamma - b^1_\gamma)\theta)}{(n_2-1)g(b^2_\gamma|n_2)}\right]}.$$

Once θ and $H(\gamma)$ are identified, we identify

$$D(\gamma) = \exp\left[-\int_\gamma^1 \frac{1}{H(t)}dt\right].$$

and $F_0(\cdot)$ by (8), as well. Moreover, $F^*(v) = D[F_0(v)]$. □

The intuition of the identification under CARA is identical to that under CRRA, though the nonlinearity of λ complicates the proof. We now end this subsection by an example where the set of valuation distributions is determined by a total variation norm.

Example 3. *For a fixed but unknown $\varepsilon \in [0,1]$, let*

$$\Gamma = \{F \in \mathcal{P} : \sup_t |F_0(t) - F(t)| \leq \varepsilon\}.$$

Let v_1 solve $F_0(v_1) = \varepsilon$, then

$$D(F_0(v)) = F^*(v) = \begin{cases} 0, & v \leq v_1 \\ F_0(v) - \varepsilon, & v \geq v_1 \end{cases}$$

Then, it is straightforward to note that any bidder with type less than v_1 will bid zero and for the rest, it will solve

$$\beta_n'(v) = \frac{[v - \beta_n(v)](n-1)f_0(v)}{F_0(v) - \varepsilon}$$

from which we can recover

$$v = b + \frac{1}{n-1}\frac{G(b|n) - \varepsilon}{g(b|n)},$$

if $v \geq v_1$. The model is not identified but if we equate the quintiles of valuations across two auctions with $n_1 < n_2$, (auction $j = 1, 2$, respectively) we get

$$b_\gamma^1 + \frac{1}{n_1 - 1}\frac{\gamma - \varepsilon}{g_1(b_\gamma^1)} = b_\gamma^2 + \frac{1}{n_2 - 1}\frac{\gamma - \varepsilon}{g_2(b_\gamma^2))},$$

leading to

$$\epsilon = \gamma - \frac{(b_\gamma^2 - b_\gamma^1)(n_1 - 1)(n_2 - 1)g_1(b_\gamma^1)g_2(b_\gamma^2)}{(n_2 - 1)g_2(b_\gamma^2) - (n_1 - 1)g_1(b_\gamma^1)}.$$

So, we can first, estimate ε and then recover pseudo-values and identify only the truncated valuation distribution.

3. Estimation

In this section, we consider semiparametric Bayesian estimation of the MEU model using experimental data used by [Dyer, Kagel, and Levin, 1989] and U.S. Timber auction data studied by [Lu and Perrigne, 2008].[12] A merit of using experimental data is that, unlike the field data but like simulation we know the true valuation distribution but at the same time unlike simulation we do not have to impose arbitrary risk preferences. If we can recover the true distribution used in the experiment and also estimate the risk aversion parameter to be similar as in [Bajari and Hortaçsu, 2005] who also use this data, we can be confident that

[12] The dataset available at `http://qed.econ.queensu.ca/jae/2008-v23.7/` archived by *the Journal of Applied Econometrics*.

when we use field data the estimates will be sensible. We begin with an estimation method then we analyze the experimental data and end with the estimation of Timber Auctions.

3.1. **Estimation Method.** We use b_j to denote the equilibrium bid of an auction with n_j bidders with $j \in \{1, 2\}$. First of all, notice that the upper bounds of the bid data, i.e., $(\bar{b}_1, \bar{b}_2)$, are parameters to be estimated. In this case, the associated statistical model is irregular and standard asymptotic distribution theory does not apply. Consider the following, widely used, example.

Example 4. *Consider $X_1, \ldots, X_n \overset{iid}{\sim}$ Uniform$[0, \theta]$ where θ is the unknown parameter. Then, the maximum likelihood estimator (MLE) for θ is given by $\hat{\theta}_{ML} = \max\{X_1, \ldots, X_n\}$, and its asymptotic distribution is a shifted exponential distribution, where the true parameter value does not belong to the interior of the support of the asymptotic distribution of the MLE. The MLE is therefore inefficient; see sections 9.4 and 9.5 of [van der Vaart, 1998].*

For such non regular cases, [Hirano and Porter, 2003] shows that the maximum likelihood estimator (MLE) is generally inefficient, but the Bayes estimator is efficient. Furthermore, the asymptotic distribution of the two-step estimator proposed in [Guerre, Perrigne, and Vuong, 2000] is yet unknown, possibly because both steps are nonparametric with estimated (pseudo) values entering the second step. For this reason, we employ a Bayesian approach. Furthermore, since identification of CRRA and CARA parameters rely on evaluating bid densities at the upper boundary $\bar{b}_j$ (Propositions 3 and 4), we need to employ a statistical model that is flexible and behaves properly at the boundaries. So we use Bernstein Polynomial Density (BPD) estimator in a Bayesian setup. [Leblanc, 2012] shows that BPD estimators have optimal mean integrated squared error and uniform bias over the *entire* support and hence have no boundary bias problem.[13]

Since we estimate the model twice, once using experimental data and again using the Timber Auction data, and the two data differ only with respect to observed auction characteristics - experimental data has no auction characteristics - we present a general framework that allows for auction covariates. We begin by introducing some notations and extension of the identification argument.

Recall that $W \in \mathcal{W} \in \mathbb{R}^L$ is a vector of observed auction covariates. The data includes W_t covariates, n_t bidders and a vector of bids $\{b_{W,t}\}$ in each $t = 1, 2, \ldots, T$.[14] Let $P_W(\cdot)$ be the marginal distribution of W and $\pi_{n|W}(\cdot|W)$ be the conditional probability mass function of n given W. We assume that the following:

[13][Petrone, 1999a,b] develop a nonparametric Bayesian method of density estimation for univariate densities based on Random Bernstein Polynomial. Given a function F on $[0, 1]$, the Bernstein polynomial of order k of F is defined by

$$B(x; k, F) = \sum_{j=0}^{k} F\left(\frac{j}{k}\right) \binom{k}{j} x^j (1-x)^{k-j}$$

and for our purpose both k and F are random.

[14]We abuse notation and use b represent both the random variable and its realization.

Assumption 8. *There is a collection of* $\{\{b_{W,t}\}, W_t, n_t\}_{t=1}^{T}$ *defined over some appropriate probability space* $(\mathcal{X}, \Sigma, \mathbb{P})$ *such that*

(1) (W_t, n_t) *are i.i.d. across auctions with respect to* $\pi_{n|W}(n|\cdot) \times P_W(\cdot)$.
(2) $P_W(\cdot)$ *is strictly increasing CDF and admits density* $p_W(\cdot)$ *that is bounded away from zero.*
(3) *The density* $\pi_{n|W}(\cdot|\cdot)$ *has finite support* $\mathcal{N}$ *and* $\pi_{n|W}(n|\cdot)$ *is strictly positive over interior of* $\mathcal{W}$.
(4) *For any pair of* $(W, n) \in \mathcal{W} \times \mathcal{N}$, *we have*

$$b_1, \ldots, b_n | W, n \overset{iid}{\sim} G(\cdot|W, n). \tag{10}$$

Now, we model $G(\cdot|W, n)$ in (10) by a semi-parametric specification, which has a full support in the space of the bid distribution while imposing a parametric structure on the dependence of b upon W. The nonparametric part of the specification is given by

$$\Psi(x|k, \omega_k) := \sum_{j=1}^{k} \omega_{j,k} \mathrm{Beta}(x|j, k-j+1), \tag{11}$$

where $\mathrm{Beta}(\cdot|a_1, a_2)$ is the Beta distribution function with parameters a_1 and a_2 with density denoted by $\mathrm{beta}(\cdot|a_1, a_2)$, and

$$\Delta_{k-1} = \left\{ (\omega_{1,k}, \ldots, \omega_{k,k}) \in \mathbb{R}_+^k \middle| \sum_{j=1}^{k} \omega_j = 1, \forall j, \omega_j \geq 0 \right\},$$

which is often called the $(k-1)$ dimensional unit simplex; $\Delta_0 = [0, 1]$ is the unit interval as a special case. Note that (11) is also called the Bernstein polynomials, and it is known that (11) forms a dense subset of the space of the distribution over Δ_0 as k increases. [Petrone, 1999a,b] develop a nonparametric Bayesian method to estimate a distribution over Δ_0 using a Dirichlet Process prior over the family of (11) with $k \in \mathbb{N}$, as we outline shortly. Let $\mathbb{1}(A)$ be the indicator for an event A. The semi-parametric model for a fixed k is then given by

$$G(b|W, n, \zeta, \eta, k, \omega_k) := \Psi\left(\tilde{G}(b|W, n, \zeta, \eta) \cdot \frac{\mathbb{1}[b \leq \tilde{G}^{-1}(p|W, n, \zeta, \eta)]}{p} \middle| k, \omega_k \right) \tag{12}$$

where $(\zeta, \eta, p) \in \mathbb{R}^L \times \mathbb{R}^L \times \Delta_0$ and $\tilde{G}(\cdot|\cdot, \cdot, \cdot, \cdot)$ is a parametric CDF that is to be further specified below depending on the application. In particular, we employ

$$\tilde{G}(b|W, n, \zeta, \eta) = \frac{b}{30} \cdot \mathbb{1}(b < 30) + \mathbb{1}(b = 30) \tag{13}$$

for the experimental data, with no auction covariates, and we use, for the timber auctions,

$$\tilde{G}(b|W, n, \zeta, \eta) := \Phi[\log(b)|W'\zeta, \exp(W'\eta)] \tag{14}$$

where $\Phi(\cdot|\mu, \sigma^2)$ be the Gaussian CDF with mean μ and variance σ^2. Note that the argument of Ψ in (12) is a truncated version of $\tilde{G}(\cdot|\cdot, \cdot, \cdot, \cdot)$, and the truncation quantile $p \in \Delta_0$ forms an estimate for the upper boundary of the bid data conditional on the covariate (n, W) and the parameter (ζ, η), i.e., $b \leq \bar{b}(W, n, \zeta, \eta) := \tilde{G}^{-1}(p|W, n, \zeta, \eta)$. The truncated parametric distribution $\tilde{G}(\cdot|\cdot,\cdot,\cdot,\cdot)$ approximates the bid distribution, $G(\cdot|W,n)$, in (10) and the nonparametric part $\Psi(\cdot|\cdot,\cdot)$ improves the approximation.

Markov Chain Monte Carlo. We explain here the Markov Chain Monte Carlo (MCMC) algorithm to draw the parameter vectors from the posterior. The data contains $\{\tilde{b}_{1,\tilde{t}},\ldots,\tilde{b}_{n,\tilde{t}},\tilde{W}_{\tilde{t}}\}_{\tilde{t}=1}^{\tilde{T}}$. For a notational convenience, we re-index all the bids and the covariates so that we have $z_T := \{b_t, W_t\}_{t=1}^{T}$ with $T = n \times \tilde{T}$, i.e., $\tilde{T}$ is the number of auctions, T is the number of total bids, and each W_t repeats n times in z_T as in the example below;

Example 5. *Suppose we observe* $\{(\tilde{b}_{1,1},\tilde{b}_{2,1},\tilde{W}_1),(\tilde{b}_{1,2},\tilde{b}_{2,2},\tilde{W}_2)\}$*, i.e.,* $(n,\tilde{T}) = (2,2)$*. Then,* $z_{T=4} := \{(b_1,W_1),(b_2,W_2),(b_3,W_3),(b_4,W_4)\}$ *with* $W_1 = W_2 := \tilde{W}_1$ *and* $W_3 = W_4 := \tilde{W}_2$ *and* $(b_1,b_2,b_3,b_4) := (\tilde{b}_{1,1},\tilde{b}_{2,1,1},\tilde{b}_{1,2},\tilde{b}_{2,2})$*.*

Let $\pi_n(\cdot,\cdot,\cdot,\cdot,\cdot)$ be the joint prior density of $(\zeta,\eta,p,k,\omega_k)$ over the parameter space $\mathbb{R}^L \times \mathbb{R}^L \times \Delta_1 \times \mathbb{N} \times \Delta_{k-1}$. This parameter space here is infinite dimensional because k is unbounded and it determines the dimensionality of Δ_{k-1}. It is, therefore, hard to draw the whole parameter vector from the posterior all at once. For this reason, we employ the Metropolis-within-Gibbs-sampler that iteratively updates the parameter vector part by part. Specifically, for a given pair of (k,ω_k), the model (12) is indexed by the parameter $(\zeta,\eta,p) \in \mathbb{R}^L \times \mathbb{R}^L \times \Delta_0$. We update (ζ,η,p) by the usual Gaussian Metropolis-Hastings algorithm regarding (k,ω_k) as given. For a given parameter (ζ,η,p), we construct the argument in (12) as follows,

$$x_t := \tilde{G}(b_t|W_t,n_t,\zeta,\eta)\cdot\frac{\mathbb{1}[b_t \le \tilde{G}^{-1}(p|W_t,n_t,\zeta,\eta)]}{p} \tag{15}$$

for all $t = 1,\ldots,T$. Then, we draw (k,ω_k) conditional on $x_1,\ldots,x_T$ and (ζ,η,p) by the algorithm of [Petrone, 1999a,b]. In order to outline the Metropolis-within-Gibbs-sampler more formally, let $(\zeta^{s-1},\eta^{s-1},p^{s-1},k^{s-1},\omega_k^{s-1})$ be the outcome of the $(s-1)$-th iteration. The algorithm, then, obtains the outcome of the s-th iteration via the following steps;

(1) draw (ζ^s,η^s,p^s) conditional on $(k^{s-1},\omega_k^{s-1};z_T,\pi_n)$ by the Gaussian Metropolis-Hastings algorithm, i.e., we draw $(\tilde{\zeta},\tilde{\eta},\tilde{p})$ from the multivariate Gaussian distribution with mean (ζ^s,η^s,p^s) and covariance Ω_s and let $(\zeta^{s+1},\eta^{s+1},p^{s+1}) := (\tilde{\zeta},\tilde{\eta},\tilde{p})$ with probability

$$\min\left[1,\frac{\pi_n(\tilde{\zeta},\tilde{\eta},\tilde{p})}{\pi_n(\zeta^s,\eta^s,p^s)}\prod_{t=1}^{T}\frac{g(b_t|W_t,n,\tilde{\zeta},\tilde{\eta},\tilde{p},k^s,\omega_k^s)}{g(b_t|W_t,n,\zeta^s,\eta^s,p^s,k^s,\omega_k^s)}\right], \tag{16}$$

otherwise, let $(\zeta^{s+1},\eta^{s+1},p^{s+1}) := (\zeta^s,\eta^s,p^s)$.

(2) draw (k^s,ω_k^s) conditional on $(\zeta^s,\eta^s,p^s;z_T,\pi_n)$ and $(x_1^s,\ldots,x_T^s)$ by Petrone [1999a,b] where x_t^s is constructed as in (15) with (ζ^s,η^s,p^s) in place of (ζ,η,p); this step is involved and we explain it in a greater detail in appendix B.

The Metropolis-within-Gibbs-sampler is a standard MCMC procedure to simulate the posterior; see [Zeger and Karim, 1991; Chib and Greenberg, 1996]. Let $\{\zeta^s,\eta^s,p^s,k^s,\omega_k^s\}_{s=1}^{S}$ be an ergodic MCMC outcome for a large S, which allows us to explore the posterior distribution. From [Tierney, 1994], we know

$$\frac{1}{S}\sum_{s=1}^{S} g(b|W,n,\zeta^s,\eta^s,p^s,k^s,\omega_k^s) \stackrel{a.s}{\to} g(b|W,n,z_T) \tag{17}$$

where the predictive bid density $g(b|W,n,z_T)$ for a given (W,n,z_T) is defined as

$$g(b|W,n,z_T) := \sum_{k\geq 1}\pi_n(k|z_T)\left\{\iiiint_{\Delta_{k-1}\times\Delta_0\times\mathbb{R}^{2L}} g(b|W,n,\zeta,\eta,p,k,\omega_k)\pi_n(\zeta,\eta,p,\omega|k,z_T)d\zeta d\eta dp d\omega\right\}$$

with $\pi_n(k|z_T)$ the posterior probability mass function of k given data z_T and $\pi_n(\zeta,\eta,p,\omega_k|k,z_T)$ the conditional posterior density function of (ζ,η,p,ω_k) given k and data z_T.[15]

Once we obtain the MCMC outcomes from the posterior densities $\pi_n(\zeta,\eta,p,k,\omega_k|z_T)$ for $n\in\{n_1,n_2\}$, we may simulate the risk aversion parameter θ and the function $D(\cdot|\cdot)$ for any given W by repeating (1) and (2) below sufficiently many times.

(1) draw

$$\zeta^j,\eta^j,p^j,k^j,\omega^j_{k^j}\sim\pi_{n_j}(\zeta,\eta,p,k,\omega_k|z_T)$$

to construct $g^j(\cdot|W) := g(\cdot|W,n_j,\zeta^j,\eta^j,p^j,k^j,\omega^j_{k^j})$ for each $j=1,2$.

(2) compute $\theta(W)$ using $g^1(\cdot|W)$ and $g^2(\cdot|W)$ and construct $D(\cdot|W)$. Note here that we only use $(\zeta^j,\eta^j,p^j,k^j,\omega^j_{k^j})$ with $j=1,2$ such that $\theta(W)$ and $D(\cdot|W)$ satisfy the theoretical shape restrictions i.e., (i) $\theta(W)\in\mathbb{R}_+$ for CARA and $\theta\in\Delta_0$ for CRRA, and (ii) $D(q|W)\leq q$ for all $q\in[0,1]$.

Since we have distributions of $\theta(W)$ and $D(\cdot|W)$, we can obtain their (posterior) predictive estimate by averaging the simulated quantities. Moreover, if W varies in the sample, we may intergrate it out with respect to the empirical distribution of W.

3.2. **Experimental Data.** The subjects were MBA students at the University of Houston. There were three experimental runs with six different subjects participating in each run, for a total of 18 subjects. In these experiments, bidders were assigned independently and identically distributed (i.i.d.) values v drawn from the uniform distribution on $[0,30]$. In the event that they won, subjects were paid their value minus their bid. Each subject participated in 28 auctions over the course of two hours. The number of bidders was determined at random in the experiment. With probability one-half, there were $n_1 := 3$ bidders and with probability one-half, $n_2 := 6$ bidders. Subjects submitted two contingent bids and one non-contingent bid. After the bids were submitted, a coin was tossed to determine whether the contingent or non-contingent bids would be used in determining the winner. A second coin toss determined whether $n=n_1$ or $n=n_2$. If the contingent treatment was selected, the first n_1 contingent bid was used if $n=n_1$, and the n_2 contingent bid was used if $n=n_2$. Otherwise, the non-contingent bid was used so that the bid could not be conditioned on

[15] The predictive bid density $g(b|W,n,z_T)$ also depends on the prior $\pi_n(\cdot)$ and the parametrization of $\tilde{G}$, which is suppressed for notational simplicity.

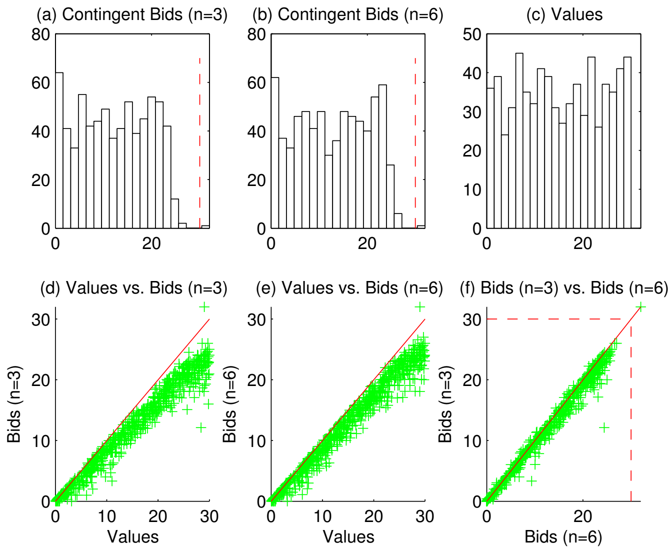

FIGURE 1. Experimental Data: Panels (a)–(c) show the histograms of contingent bids for $n_1 = 3$, $n_2 = 6$, and true values (drawn from $U[0,30]$) along with the upper bound of the value (30) by a dashed vertical line. Panels (d) and (e) are the scatter diagrams of values against bids along with the 45 degree line, and panel (f) does similarly between bids n_1 and n_2 along with the 45 degree line and the marks for the upper bound of values.

n. After each auction, bids and corresponding private values were publicly posted on a blackboard.

We only use the contingent bids; see Figure 1.[16] Panels (a)–(c) show the histograms of contingent bids for n_1, n_2, and true values. The vertical dashed lines on panels (a) and (c) are the upper bound of the value. Panels (d) and (e) are the scatter diagrams of values against bids along with the 45 degree line, and panel (f) does similarly between bids n_1 and n_2 along with the 45 degree line and the marks for the upper bound of values. Total number of values (thus, the number of bids for each n) is 705.

Some features of the data suggest that the subjects deviate from the usual bidding strategy. First, four subjects choose to obtain nonpositive utility: among them, two bid zero while their values are strictly positive (say, 0.1 and 0.5) and two other bid higher than or equal to their own values for both n_1 and n_2, e.g., Figure 1 shows that a subject bid 32 for both n's, whereas his or her value was 29.02. Second, 33.5 % of the subjects bid higher for auctions with n_1 than the auctions with n_2, i.e., $b_1 \geq b_2$, whereas it must be that $b_1 < b_2$ because $n_1 < n_2$. Third, subjects tend to overbid. In particular, 87.1% of the subjects bid higher, in the auctions with n_1 bidders, than the equilibrium bid with no risk aversion under EU, and 61.4% did in the auctions with n_2 bidders.

[16] For detailed analysis on the data see [Bajari and Hortaçsu, 2005].

	Mean	St. Dev	2.5^{th}	Median	97.5^{th}
θ_{CRRA}	0.9045	0.0337	0.8202	0.9110	0.9515
θ_{CARA}	0.5898	0.2013	0.2600	0.5600	1.0350
$\bar{b}_{N=3}$	26.3319	0.2758	26.0385	26.2725	27.0706
$\bar{b}_{N=6}$	27.4371	0.3295	27.0387	27.3641	28.2746

TABLE 1. Point Estimates: The posterior mean (the posterior standard deviation) of the parameters of risk aversion and the upper boundaries of the bid distributions, where q^{th} is the q^{th} percentile.

Note: The subjects in this experiment were told the values were drawn *i.i.d* from uniform distribution, but that does not mean we cannot use the data to estimate model with ambiguity. It is our view that if there is no distinction between "being told" and "knowing" the distribution, it will be reflected in the estimation and we will reject presence of ambiguity. Moreover, this experiment was not designed for ambiguity and any finding should be interpreted within the context of this model.

3.2.1. *Estimation Results.* Table 1 shows the posterior means (the posterior standard deviations in parentheses) of the parameters $(\theta_{crra}, \theta_{cara})$ and the upper boundaries $(\bar{b}_{N=3}, \bar{b}_{N=6})$. What is interesting and important to note is that [Bajari and Hortaçsu, 2005]'s estimate of the CRRA parameter (using the same data) is contained within the 95% credible band $[0.8202, 0.9515]$ of our estimate. This means that even though our model is new in the literature, it matches the estimates in the literature.

We are also interested to see if $D(\cdot)$ can be reasonably approximated by the 45° line. If it can be, we conclude that there is no evidence for ambiguity, but otherwise we conclude that there is evidence for ambiguity. To that end, we compute the posterior distribution of $\|D(\gamma) - \gamma\|_2$.[17] The mean, standard deviation, 2.5^{th} percentile, median and 97.5^{th} of posterior distribution of the L_2 distance for CRRA and CRRA utilities, respectively are given in Table 2. As can be seen the estimates supports the hypothesis that there is ambiguity, i.e. $D(\cdot)$ is indeed different from identity mapping.

The estimation results are also presented in Figure 2. The first row contains the histogram of bids along with the posterior predictive bid densities $g(\cdot|n)$ and a 95% credible band for $n = 3$ and 6, respectively. The bids are normalized to be between $[0, 1]$ by the factor of the estimate of the bid upper bounds. The second row contains the point-wise mean of $D(\cdot)$ (the solid line) with its 95% credible band (dashed), the distribution of L_2-norm $\|\cdot\|_2$ to measure the distance between $D(\cdot)$ and the 45° and the posterior distribution of the risk aversion parameter, all for CRRA utility in Panels (c)-(e) respectively. Similarly the third row contains the same thing for CARA utility. As can be seen the visualization supports the hypothesis that there is ambiguity, i.e. $D(\cdot)$ is indeed different from identity mapping.

[17]The L_2 distance of the 45° line from the horizontal axis $\|\gamma - 0\|_2$ is equal to 0.5774.

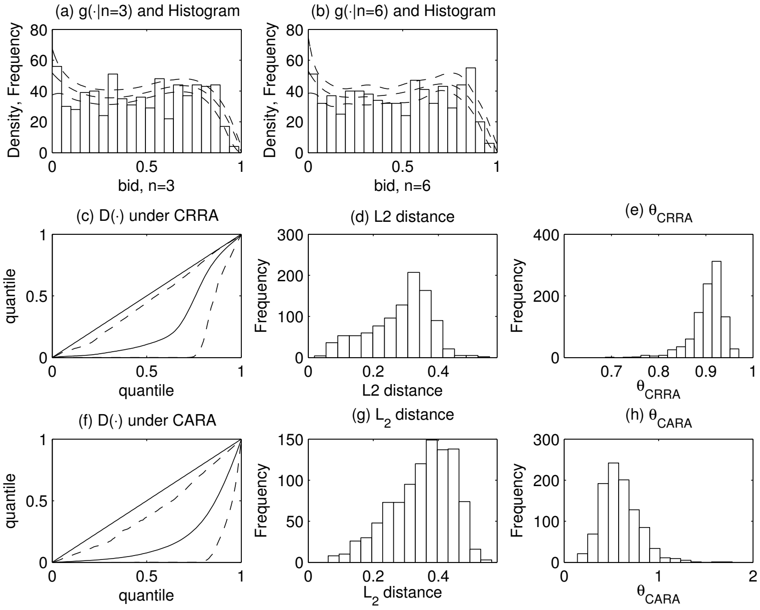

FIGURE 2. Estimation results: Panels (a) & (b) show the histogram of the data, where the middle dotted curves is the posterior predictive density of $g(\cdot|n)$ with its 95% credible band, for $n = 3$ and $n = 6$, respectively. Panel (c) & (f) is the (nonparametrically) estimated $D(\cdot)$ by its point-wise mean (solid) along with its 95% credible band (dashed); Panels (d) & (g) are the distribution of L_2 distance between estimated $D(\cdot)$ and the 45° line; and Panel (e) & (g) are the posterior distribution of the risk aversion parameter, for CRRA and CARA specifications, respectively.

	Mean	St. Deviation	2.5^{th}	Median	97.5^{th}
CRRA	0.2807	0.0935	0.0857	0.0987	0.1310
CARA	0.3508	0.0953	0.1381	0.1712	0.2156

TABLE 2. Posterior Probability of $\|D(\gamma) - \gamma\|_2$, where q^{th} is the q^{th} percentile.

3.3. **Timber Auctions.** As mentioned earlier, the fact that our estimates of CRRA coefficient is almost identical to that of [Bajari and Hortaçsu, 2005] and the estimated valuation distribution is uniform like the true data generating process, reassures that our estimate for ambiguity is reasonable. It is also desirable to know if similar results would hold for field data where the bidding procedure is more complex than in the experiment. To answer that question in this section, we study a sample from the sealed bid U.S timber auctions data that has been widely used in the literature. We begin by explaining the data, then we show how we can exploit the special feature of the data to achieve nonparametric identification and finally estimate the model.

3.3.1. *Data.* The U.S. Forest Services sells the timber from publicly owned forests. We use the auction data organized in the western half of the U.S. (regions 1-6) in 1979. The data contains information about the the identity and the average bid for every tract of each participating bidder, detailed information on the estimated total volume (in mbd) and appraisal value (per unit of volume, in dollar) of timber by the Forest Service, the season during which the auction was held and the exact location of the timber parcel. characteristics of the lot from the Forest Service sale announcement. In this paper we focus on auctions with $n_1 = 2$ and $n_2 = 3$ and for identification we assume that these numbers are exogenous.[18] As shown by [Lu and Perrigne, 2008] reserve prices are too small and nonbinding, so we too ignore them.[19] The most interesting feature about the data is the way in which the bidders gather information about a tract. First the Forest Service officials "cruise" the selected tract of timber and estimate the quantity of each spices, which are then publicly announced. Then the bidders may conduct their own cruises to form their own estimates. Cruising a tract is considered something of an art by industry experts and bidders either have in-house experts in forestry or hire some experts in the market with varying degree of experiences and skills; see [Athey and Levin, 2001; Athey, Levin, and Seira, 2011] for more. Therefore, it seems plausible that although a bidder can determine the worth of the tract knowing the exact distribution of opponents' appraisal is at least questionable and should be verified from the data.

There are $T_1 = 107$ auctions with n_1 bidders, for which the average (standard deviation) of the bids is 88.50 (52.91), and the minimum and maximum are 2.78 and 295.03, respectively. In Figure 3, panels (a) and (b) are the scatter plots of the bids against the logarithms of total volume and bids against the log of appraisal value per unit, respectively for auctions with two bidders. Similarly, panels (c) and (d) are for auctions with three bidders. The correlation coefficient between the winning (losing) bid and the total volume is -0.15 (-0.17), and the one between the winning (losing) bid and the appraisal value is 0.72 (0.78). There are $T_2 = 108$ auctions with n_2 bidders, for which the average (standard deviation) of the bids is 84.44 (52.63) and the minimum and maximum are $(1.05, 328)$. The correlation coefficient between the winning (2nd,3rd) bid and the total volume is -0.18 (-0.16,-0.25), and the one between the winning (2nd,3rd) bid and the appraisal value is 0.64 (0.69,0.77).

3.3.2. *Estimation Results.* The analog of Figure 2 for Timber auction is Figure 4. Since there are 215 different auctions, we estimate each parameter evaluated at each $W_t, t = 1, \ldots, 215$. The first column corresponds to posterior mean $D(\cdot|W_t)$ evaluated at $W_t, t = 1, \ldots, 215$. As can be seen, $D(\cdot|\cdot)$ is significantly different from the identity mapping, suggesting ambiguity for both CARA and CRRA specifications. In column two contains the density of L_2- distance between $D(\cdot|\cdot)$ and 45° line and column three is the posterior densities of the risk aversion parameter. These densities show that the risk aversion parameter for CRRA is concentrated around 0.5 while the CARA coefficient is close to 0, i.e. risk neutral. This estimates are consistent with [Lu and Perrigne, 2008], which shows we do as good as the

[18] [Aradillas-López, Gandhi, and Quint, 2011] show that this exogenous participation assumption is valid for Timber auction.

[19] Our estimation method can be extended to allow for binding reserve price, something which we do not pursue in this paper in view of the objective.

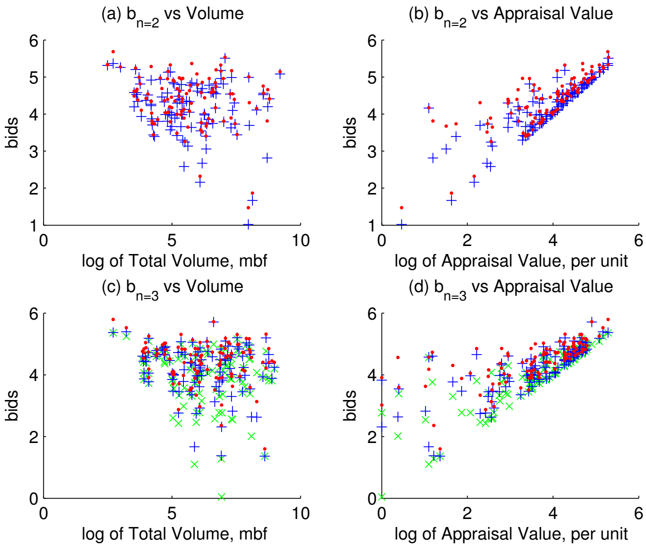

FIGURE 3. Scatter Plot of Timber Auction: Panels (a) and (b) show the bids and log of total volume and bids and log of appraisal value per unit, respectively for auctions with $n = 2$ bidders; and Panels (c) and (d) are for auctions with $n = 3$ bidders. Legend: '$\cdot$' (red) is the winning bid and '$+$' (blue) is the 2^{nd} highest bid and 'x' (green) is the 3^{rd} highest bid.

		Minimum	2.5^{th}	5^{th}	Median	Mean	Std. Dev.
	minimum	0.0042	0.03257	0.0619	0.3955	0.3737	0.1160
CRRA	median	0.0483	0.0864	0.1204	0.4451	0.4076	0.1406
	maximum	0.1329	0.1736	0.2201	0.4945	0.4540	0.1674
	minimum	0.0057	0.0507	0.0868	0.4397	0.4041	0.0981
CARA	median	0.0632	0.1121	0.1691	0.4870	0.4433	0.1281
	maximum	0.1696	0.2339	0.2908	0.5251	0.4736	0.1557

TABLE 3. Posterior Probabilities of $\|D(\gamma) - \gamma\|_2$: The summary statistics of posterior probability of the L_2 distance between $D(\cdot)$ and 45° line, where the minimum, median and the maximum are across 215 auctions.

previous literature in terms of risk aversion but we also estimate ambiguity that is novel. A further evidence that the L_2- distance is strictly bounded away from zero is presented in Table 3. Under CRRA specification we compute different moments from the posterior distribution of L_2- distance, one for each $W_t, t = 1, \ldots, 215$ and for each such moment we compute the mean, median and the maximum across all covariates W_t. We do the same for CARA specification as well. From Table 3 we can see under CRRA specification, the median of the distance ranges from 0.3955 to 0.4945 with a median at 0.4451. Similarly, for CARA it ranges between 0.4397 and 0.5251 with a median at 0.4870. The minimum distance across all auctions is 0.0042 for CRRA and 0.0057 for CARA. We can safely conclude that there is evidence of ambiguity in the data.

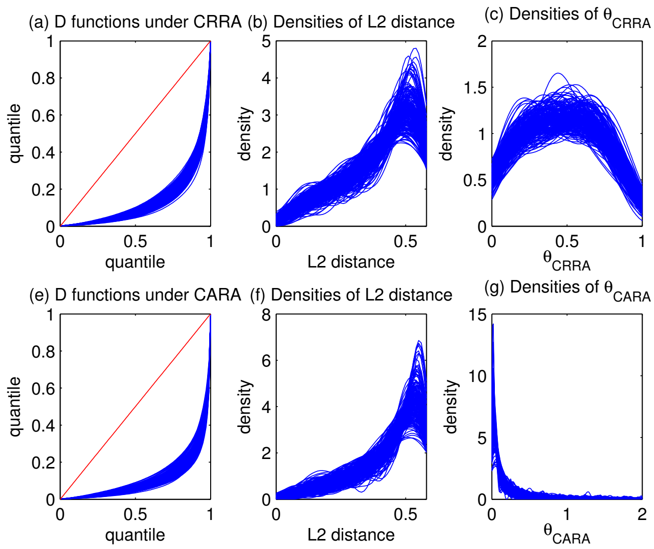

FIGURE 4. Estimation Results (Timber Auctions): Panels (a) & (e) are the (nonparametrically) estimated $D(\cdot|W)$ by its point-wise mean evaluated; Panels (b) & (f) are the nonparametric density of the L_2- distance between $D(\cdot|\cdot)$ and 45° line; and Panels (c) & (g) are the posterior distribution of the risk aversion parameter, for CRRA and CARA specification. All of these estimates are evaluated at 215 different auction covariates W observed in the data.

4. EXTENSION: UNOBSERVED HETEROGENEITY

In this section we consider auctions auctions with nonspearable unobserved heterogeneity. As [Krasnokutskaya, 2011] showed, some auctions have traits that are observed by bidders and affect the valuation but are unobserved by the econometricians. Failure to incorporate such traits in estimation adversely effect the inference. Therefore we augment our model to allow for unobserved heterogeneity that is *nonseparable*.[20]

Let $Y \in [\underline{y}, \overline{y}] \subset \mathbb{R} \overset{i.i.d}{\sim} P_Y(\cdot)$ denote those auction specific traits that are observed only by the bidders such that $v|Y = y \overset{i.i.d}{\sim} F_0(\cdot|y)$. For every auction the realization of Y is common knowledge among all bidders and let the set of (conditional) valuation distributions be Γ_Y, which now depends on Y. Everything else defined earlier works as long as we consider conditional distributions. We begin with the following assumption:

Assumption 9. *(Monotonicity) For all $y' > y$, $F_0(\cdot|y') \leq F(\cdot|y)$ for all v_i with strict inequality for some.*

This is a strong but important assumption that guarantees existence of symmetric equilibrium. Since bidders observe Y and values are conditionally independent the observed bids are also conditionally independent. In other words, let b_i be the bid of bidder i and $G(\cdot|n, y)$ be conditional bid distribution, then $(b_1, \ldots, b_n)|Y = y \sim$ i.i.d. $G(\cdot|n, y)$, with conditional

[20][Krasnokutskaya, 2011] considers $v_i = \tilde{v}_i \times Y$, where $\tilde{v}_i$ is the individual value.

density $g(\cdot|n, y)$. Let $\overline{b}(y)$ be the highest bid in the auction with $Y = y$. Then we follow [Hu, McAdams, and Shum, 2011] and make the following technical assumptions.

Assumption 10. *Unobserved Heterogeneity*

(1) *The functional* $\max supp(b_i|y)$, *the maximal of the equilibrium bid support be is strictly increasing in* y.
(2) *For all* i, k *and* $s : b \to \mathbb{R}$ *with bounded conditional expectation, if* $\mathbb{E}[s(b_k)|b_i] = 0$ *for all* $b \in [0, \overline{b}(\underline{y})]$, *then* $s(b_k) = 0$ *for all* $b_k \in [0, \overline{b}(\underline{y})]$.
(3) *For all* i *and* $s : [\underline{y}, \overline{y}] \to \mathbb{R}$ *with bounded conditional expectation, if* $\mathbb{E}[s(Y)|b_i] = 0$ *for all* $b_i \in [0, \overline{b}(\underline{y})]$, *then* $s(y) = 0$ *for all* $y \in [\underline{y}, \overline{y}]$.

Part 1 of the assumption can be shown to follow from assumption 9. The remaining two (technical) assumptions are the usual completeness assumptions.[21] We state the main identification result of this subsection.

Lemma 3. *Let assumptions 1- 4, 9 and 10 be satisfied and all auctions have at least three bidders. Then*

(1) *if assumption 5 holds, then the CRRA parameter* θ,
(2) *or if assumption 6 holds the CARA parameter* θ

and the conditional valuation distribution $F_0(\cdot|Y)$, *the function* $D(\cdot)$ *and the marginal distribution of* $P(\cdot)$ *are nonparametrically identified by* $g_1(\cdot|n_1)$ *and* $g_2(\cdot|n_2)$ *with* $n_2 > n_2$.

Sketch of the proof: From [Hu and Schennach, 2008; Hu, McAdams, and Shum, 2011] we can identify the joint cdf of $(b_1, \ldots, b_n, Y)$ from the cdf of $(b_1, \ldots, b_n)$ and hence the conditional distribution $G(\cdot|n, y)$ and the marginal distribution $P_Y(\cdot)$. Repeating this for two auctions with n_1 and n_2 bidders we identify $G_1(\cdot|n_1, y)$ and $G_2(\cdot|n_2, y)$ and their corresponding densities. Since the equilibrium strategy is still strictly increasing, there is a one-to-one mapping between bids and valuation, given Y, identification of the structural parameters follows exactly like before.[22]

5. Multiplier Preferences

[Hansen and Sargent, 2001] considers a situation where decision makers, i.e., bidders, have an initial estimate of the true distribution, but are worried about a misspecification of it, and consider other distributions that are not too far away in terms of relative entropy.[23] Within our environment, the initial estimate will be the true distribution $F_0(\cdot|n)$, suppressing the dependence on observed covariates ω unless mentioned otherwise, and any departure (in terms of the bidding behavior) from it will be attributed to ambiguity.

[21] These assumptions are widely used in the literature on nonparametric instrument variables [Newey and Powell, 2003; D'Haultfoeuille, 2011] and in nonlinear measurement error models [Schennach, 2004]. If Y is finite or when the model is linear then the assumptions are equivalent to the widely used rank conditions.

[22] Estimation of this auction model, however, is an open question; see [Schennach, 2004] for more on this.

[23] Let $K(F\|F_0)$ is the Kullback-Leibler divergence (or the relative entropy) of $F(\cdot|n)$ with respect to the true distribution $F_0(\cdot|n)$.

Assumption 11. *The preference order of each bidder satisfies assumptions A1-A6, A8 and P2 in [Strzalecki, 2011].*

If there are at least three disjoint non-null events, then assumption 11 is necessary and sufficient for each bidder's preferences to have the multiplier preference representation, so that a bidder with value v solves

$$\max_{x\in[\underline{v},\overline{v}]} \min_{F\in\mathcal{P}_n} \left\{ u[v-\beta_n(x)]F(x|n)^{n-1} + \alpha K(F\|F_0) \right\}. \tag{18}$$

Here $\alpha \in (0,\infty]$ captures the bidders' confidence on their initial estimate $F_0(\cdot|n)$, and can be thought of as the degree of ambiguity. For instance, if $\alpha = \infty$, the minimization is solved by $F_0(\cdot|n)$ with the interpretation that the bidders are certain that $F_0(\cdot|n)$ is the true distribution. It is also known that (18) is equivalent to

$$\max_{x\in[\underline{v},\overline{v}]} \Lambda_\alpha \left(u[v-\beta_n(x)]\right) F_0(x)^{n-1} \tag{19}$$

where

$$\Lambda_\alpha(t) := 1 - \exp\left(-\frac{t}{\alpha}\right) \tag{20}$$

where $\alpha > 0$ in assumption 6; see appendix A and also [Strzalecki, 2011]. Since (20) is a concave function, the MP representation for ambiguity averse bidders is equivalent to the EU representation for *more* risk averse bidders without ambiguity, i.e., $U(\cdot) := \Lambda_\alpha \circ u(\cdot)$. Intuitively, therefore, the bidders attitude toward risk and that for ambiguity cannot not be separately identified;

Proposition 5. *Under assumptions 1, 2, 3, 4 and 11, the distribution function F_0 is nonparametrically identified but the utility function u is identified up to a multiplicative constant α by the knowledge of bid distributions $g(\cdot|n_1)$ and $g(\cdot|n_2)$ where $n_1, n_2 \in \mathcal{N}$ with $n_1 < n_2$. Moreover, if bidders are risk neutral, α is identified by $g(\cdot|n_1)$ and $g(\cdot|n_2)$.*

Proof. By [Guerre, Perrigne, and Vuong, 2009], U and F_0 are nonparametrically identified by $g(\cdot|n_1)$ and $g(\cdot|n_2)$. From (20), we have $u(t) = -\alpha \log[1-U(t)]$. Hence, if bidders are risk neutral, i.e., $u(t) = t$, the MP parameter α is identified, i.e., $\alpha = -t/\log[1-U(t)]$. □

As proposition 5 indicates, if bidders are risk neutral, the ambiguity parameter under MP is given by

$$\alpha = -\frac{t}{\log[1-U(t)]}.$$

The fact that RHS is a constant provides a testable restriction. When bidders are risk averse, however, the utility function is not identified as MP is equivalent to EU with more risk averse bidders. Though MP is observationally equivalent to EU, it is important to consider the bidders' attitude toward ambiguity, i.e., α, separately from that toward risk. Failure to identify α would force an investigator to choose the parameter arbitrarily.[24]

The structure of ambiguity, however, affects the optimal auction design just like under MEU [Bose, Ozdenoren, and Pape, 2006; Bodoh-Creed, 2012]. So it might be desirable to find other plausible condition that identifies α even when bidders are risk averse. One such condition is to normalize one quantile of the valuation distribution. For illustration, suppose the highest value $\overline{v}$ is a either known to the researcher or just a function of n and the observed auction covariates W.[25] Then, the optimality condition gives

$$u(\overline{v} - b(1)) = \frac{1}{\alpha} \ln\left(1 + \frac{\alpha}{(n-1)g(b(1)|n, W)}\right),$$

where everything except α is known, and has a unique solution in α because the RHS is strictly decreasing and convex in α.

Lemma 4. *Under assumptions 1, 2, 3, 4 and 11, the distribution function $F_0(\cdot|n, W)$, the utility function $u(\cdot)$ and the ambiguity parameter α are (nonparametrically) identified from $g(\cdot|n_1, W)$ and $g(\cdot|n_2, W)$ where $n_1, n_2 \in \mathcal{N}$ with $n_1 < n_2$, if in addition γ quantile $v(\gamma; n, W) = v(\gamma; n, W, \tau)$ for all $\gamma \in (0, 1]$ where $\tau \in \mathbb{R}^{d'}$ is a d' dimensional unknown parameter.*

An application of this result would be to estimate this model and do a model selection between MEU and MP, since there is no guidance in the literature about which of the many representations should be used for empirical analysis. We leave this line of inquiry for future research.

6. Conclusion

In this paper we analyze first-price auction where (risk averse) bidders are ambiguous about the valuation distribution. We depart from classic EU where it is assumed that the true distribution is commonly known by all bidders and instead allow bidders to consider a set of distributions as likely candidate for valuation distribution, and determine an equilibrium in symmetric and monotonic bidding strategies under MEU representation. Using variation in the number of bidders across auctions we show the valuation distribution and a function that governs ambiguity can be nonparametrically identified besides the risk aversion (CRRA or CARA) coefficient. In the process we also propose simple and implementable test for ambiguity in data. We propose a semiparametric Bayesian estimation procedure and use it to study an experimental auction and USFS timber auctions and conclude that estimates in both data show evidence of ambiguity.

It is shown that the identification strategy extends even to auctions with non separable unobserved heterogeneity as long as there are at least three bidders in each auction. We

[24] "...policy recommendations based on such a model would depend on a somewhat arbitrary choice of the representation. Different representations of the same preferences could lead to different welfare assessments and policy choices, but such choices would not be based on observable data."–[Strzalecki, 2011].

[25] We can estimate this function from the data as long as we know the functional form, but for the identification argument we assume we know the function.

also consider ambiguity with multiplier preference and show that while the identification of the utility function follows from [Guerre, Perrigne, and Vuong, 2009], the parameter that governs ambiguity is identified if we normalize a quantile of the distribution, such as the highest value. We do not estimate this model because it requires us to estimate nonparametric utility function, which is beyond the scope of this paper. Nonetheless, using our identification argument one could follow [Kim, 2013] to estimate the nonparametric utility function, which can then be used to shed light on which of the two, maxmin or the multiplier preferences, explains the data better.

Some other potentially important applications of our model includes T-bill auctions during a financial crisis, electricity market with wind turbine generators which makes the supply of electricity uncertain and auction of a new product, etcetera. Finally, allowing affiliation among bidders is also important from the point of view of empirical auction design, and left as a future research topic.

Appendix A. Multiplier Preferences

In this section, we show that (18) is equivalent to (19). The purpose of this section is to convince readers about the equivalence relation between (18) and (19). It exploits a result that is already known in the literature of decision theory; see [Strzalecki, 2011]. From (18), we have

$$\begin{aligned}
\min_{F\in\Gamma}\left\{\int u dF + \alpha K(F\|F_0)\right\} &= \alpha \min_{F\in\Gamma}\left\{\int \frac{u}{\alpha} dF + K(F\|F_0)\right\} \\
&= \alpha \min_{F\in\Gamma}\left\{\int \frac{u}{\alpha} dF + \int \log\left(\frac{dF}{dF_1}\cdot\frac{dF_1}{dF_0}\right) dF\right\} \\
&= \alpha \min_{F\in\Gamma}\left\{\int \frac{u}{\alpha} dF + K(F\|F_1) - \int \log\left(\frac{dF_1}{dF_0}\right) dF\right\}.
\end{aligned}$$

The second equality holds because for all $F \in \Gamma$ we have $F \gg F_0$ and $F_0 \gg F$. Let F_1 be the candidate equilibrium such that

$$\frac{dF_1}{dF_0} := \frac{e^{-u/\alpha}}{\int e^{-u/\alpha} dF_0}.$$

Using this definition of F_1, the RHS in the last equality become

$$\begin{aligned}
&\alpha \min_{F\in\Gamma}\left\{\int \frac{u}{\alpha} dF + K(F\|F_1) - \int \frac{u}{\alpha} dF - \log\int \exp\left(-\frac{u}{\alpha}\right) dF_0\right\} \\
= \;& \alpha \min_{F\in\Gamma}\left\{K(F\|F_1) - \int \log\left[\int \exp\left(-\frac{u}{\alpha}\right) dF_0\right] dF\right\} \\
= \;& \alpha \min_{F\in\Gamma}\left\{K(F\|F_1) - \log\left[\int \exp\left(-\frac{u}{\alpha}\right) dF_0\right]\right\}.
\end{aligned}$$

We can then see that the second term is independent of F and the minimum is achieved when $K(F\|F_1) = 0$, which happens if and only if $F = F_1$. Therefore, F_1 defined above solves the minimization problem as desired and the minimum is equal to

$$-\alpha \log\left[\int \exp\left(-\frac{u}{\alpha}\right) dF_0\right],$$

and in auction the utility is $u(v-b)$ if the bidder wins the auction and $u(0)=0$, otherwise. Therefore, (18) can be written as

$$\begin{aligned}
& \max_{z\in[\underline{v},\overline{v}]} -\alpha \log\left\{\exp\left[-\frac{u(v-\beta_n(z))}{\alpha}\right] F_0(z)^{n-1} + \exp(0)\left[1-F_0(z)^{n-1}\right]\right\} \\
= & \min_{z\in[\underline{v},\overline{v}]} \left\{\exp\left[-\frac{u(v-\beta_n(z))}{\alpha}\right] F_0(z)^{n-1} + \left[1-F_0(z)^{n-1}\right]\right\} \\
= & \max_{z\in[\underline{v},\overline{v}]} \left\{1-\exp\left[-\frac{u(v-\beta_n(z))}{\alpha}\right]\right\} F_0(z)^{n-1} \\
= & \max_{z\in[\underline{v},\overline{v}]} \Lambda_\alpha\left(u[v-\beta_n(z)]\right) F_0(z)^{n-1},
\end{aligned}$$

which (19) as desired.

Appendix B. The Nonparametric Method of [Petrone, 1999a,b]

In this section, we explain the step (2) in the Metropolis-within-Gibbs-sampler. For the specification in (11), Petrone [1999a,b] propose a class of prior distributions that has full support over $\mathcal{P}$, the set of all distributions with support Δ_0 and can easily select an absolutely continuous distribution function with a continuous and smooth derivate. Since the algorithm is based on Dirichlet distribution and Dirichlet process, we introduce these concepts first; see Ferguson [1973] for a thorough treatment. Let $Dir(a_1,\ldots,a_k)$ be the Dirichlet distribution over Δ_{k-1} characterized by parameter vector $a \in \mathbb{R}^k_+$. Consider $\omega_k \sim Dir(a_1,\ldots,a_k)$ and let $a_0=\sum_{j=1}^k a_j$. Then,

$$E[\omega_{j,k}] = \frac{a_j}{a_0},\ V[\omega_{j,k}] = \frac{a_j(a_0-a_j)}{a_0^2(a_0+1)}, \text{ and } Cov[\omega_{i,k},\omega_{j,k}] = -\frac{a_i a_j}{a_0^2(a_0+1)} \text{ for } i\neq j. \tag{21}$$

Then the Dirichlet Process over $\mathcal{P}$ is characterized by parameters $\alpha_0\in\mathbb{R}_+$ and $P_0\in\mathcal{P}$ and is denoted by $DP(\alpha_0 P_0)$. Consider $P\sim DP(\alpha_0 P_0)$, then

$$P(A_{1,k}),\ldots,P(A_{k,k}) \sim Dir[\alpha_0 P_0(A_{1,k}),\ldots,\alpha_0 P_0(A_{k,k})] \tag{22}$$

for any $k\in\mathbb{N}$ and any partition $A_{1,k},\ldots,A_{k,k}$ of $[0,1]$, i.e. $\cup_k A_{ik}=[0,1]$ and $A_{ik}\cap A_{i'k}=\emptyset$ for all $i\neq i'$. Using (21) the moments become $E[P(A_{j,k})]=P_0(A_{j,k})$; $Var[P(A_{j,k})] = \frac{P_0(A_{j,k})[1-P_0(A_{j,k})]}{\alpha_0+1}$ and $Cov[P(A_{i,k}),P(A_{j,k})] = -\frac{P_0(A_{i,k})P_0(A_{j,k})}{\alpha_0+1}$ for all $i\neq j$. Note that the DP is indexed by its mean P_0 and precision α_0.

Consider as data generating process (DGP), like Petrone [1999a], that

(1) draws $k\sim\pi_k(\cdot)$ and $P|k\sim DP(\alpha_0 P_0)$; and
(2) draws $x_1,\ldots,x_T \overset{iid}{\sim} H(\cdot|k,\omega_k)$ in (11) where $\omega_{j,k} := P\left(A_{j,k}\right)$ with $A_{j,k} := \left(\frac{j-1}{k},\frac{j}{k}\right]$.

Here, $x_1,\ldots,x_T$ are observed and $[\pi_k(\cdot),P_0(\cdot),k]$ represent the prior beliefs. Moreover, the prior beliefs can be equally represented by $[\pi_k(k),\pi_{\omega_k|k}]$ where $\pi_{\omega_k|k}$ is the density of

$Dir[\alpha_0 P_0(A_{1,k}),\ldots,\alpha_0 P_0(A_{k,k})]$.[26] To develop a tractable MCMC algorithm, Petrone [1999a] exploits the fact that this DGP is equivalent to the one that

(1) draws $k \sim \pi_k(\cdot)$ and $P|k \sim DP(\alpha_0 P_0)$;
(2) draws $y_1,\ldots,y_T \overset{iid}{\sim} P|k$; and
(3) for $t = 1,\ldots,T$, draw independently $x_t|y_1,\ldots,y_T,k,P \sim \text{Beta}(\cdot|j_t, k-j_t+1)$ where j_t is such that $y_t \in A_{j_t,k}$, i.e., $j_t = \lceil y_t k \rceil$, the smallest integer greater than $y_t k$.

This DGP draws the latent label $y_1,\ldots,y_T$ from a random probability measure P to pick a bin $A_{j_t,k}$ for each t, and draws x_t from the 'smoothed' histogram, $\text{Beta}(\cdot|j_t, k-j_t+1)$, over $A_{j_t,k}$.[27] Then to draw k, ω_k from the posterior distribution conditional on the sample $x_1\ldots,x_T$ and the prior $[\pi_k(\cdot), \pi_{\omega_k|k}(\cdot|\cdot)]$ we follow Petrone [1999a] and use the following algorithm:

(1) Draw k from the probability mass function proportional to
$$\pi_k(k)\prod_{t=1}^{T}\text{Beta}[x_t|\lceil y_t k\rceil, k-\lceil y_t k\rceil+1].$$
(2) Let $\omega^0_{j,k} := P_0(A_{j,k})$ with $A_{j,k} := \left(\frac{j-1}{k},\frac{j}{k}\right]$. For every $t = 1,\ldots,T$, let $y_t = y_{\tilde{s}}$ with probability $q_{t,\tilde{s}} \propto \text{beta}\,(x_{\tilde{s}}|\lceil y_{\tilde{s}}k\rceil, k-\lceil y_{\tilde{s}}k\rceil+1)$ for all $\tilde{s} \in \{1,\ldots,T\}\setminus\{t\}$, and, otherwise, draw y_t from a density proportional to
$$p_0(y)\text{beta}\,(x_t|\lceil yk\rceil, k-\lceil yk\rceil+1)$$
with probability $q_{t,0} \propto \alpha_0 h(x_t|\omega^0_k, k)$. Note that $\sum_{\tilde{s}=0}^{T} q_{t,\tilde{s}} = 1$ with $q_{t,t} = 0$.
(3) Let r_k be the k dimensional vector where each j-th element counts the number of y_s's in the interval $\left[\frac{j-1}{k},\frac{j}{k}\right]$, and finally $\omega_k \sim Dir[\alpha_0(\omega^0_k + r_k)]$.
(4) Iterate steps 1–3, until convergence.

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[26] If $\pi_k(k) > 0$ for all $k > 1$, the DGP is nonparametric.
[27] $B(\cdot|j_t, k-j_t+1)$ has its mode at $\frac{j_t-1}{k-1}$.

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