



Beliefs and Public Good Provision with Anonymous Contributors

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Beliefs and Public Good Provision with Anonymous Contributors*

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Abstract

We analyze a static game of public good contributions where finitely many anonymous players have heterogeneous preferences about the public good and heterogeneous beliefs about the distribution of preferences. In the unique symmetric equilibrium, the only individuals who make positive contributions are those who most value the public good and who are also the most pessimistic; that is, according to their beliefs, the proportion of players who value the most the public good is smaller than it would be according to any other possible belief. We predict whether the aggregate contribution is larger or smaller than it would be in an analogous game with complete information (and heterogeneous preferences), by comparing the beliefs of contributors with the true distribution of preferences. A tradeoff between preferences and beliefs arises if there is no individual who simultaneously has the highest preference type and the most pessimistic belief. In this case, there is a symmetric equilibrium, and multiple symmetric equilibria occur only if there are more than two preference types.

Key words: belief; contribution; free rider; public good; symmetric equilibrium.

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1 Introduction

Paul Samuelson pioneered the analysis of public goods provision.¹ More recent literature on the provision of public good under incomplete information usually assumes that incompleteness arises from either ignorance about players' preferences (Bac [2], Menezes et al. [8], Bag and Roy [3]) or from ignorance about opponents' wealth (Andreoni [1], Grandstein [5], Grandstein et al. [6]). However, in practice, it is unlikely that each individual knows the identity of all her many opponents. Even when each player knows all identities, we may not wish to assume that each player is capable of forming a joint distribution about the profiles of types for her opponents. It seems more reasonable that each player simply has a general perception of the distribution of relevant characteristics of the population.

Hellwig [7] introduces a general mathematical framework to work with incomplete information games in a large population of *anonymous* players.² This is a convenient modeling setup in games where only the aggregated action of the opponents is the relevant variable for a player. Examples of this framework are voting games, Cournot competition and public contribution games.

Our work analyzes a static game of individual contributions to a continuous public good, in a finite population of heterogeneous anonymous players. The heterogeneity comes from two sources: (i) the preferences for, or values of, the public good – what we refer to as the player's type or preference type, and (ii) the beliefs about the distribution of types along the population. Unlike most of the literature, we do not assume that each player knows the correct joint distribution of her opponents' (preference) types. Instead, each player has a belief about how the population is divided by types. Each individual has quasi-linear utility in her private good consumption and chooses how much of her initial

¹See Samuelson [11] and [12], as well as Olson [9].

²As Hellwig [7] explains, "*This formulation of incomplete information departs from the personalistic approach where each agent is assumed to receive a signal about the underlying state of the world and uses the signal to form probabilistic beliefs about the state of the world, about the signals received by the other agents and about the other agents' beliefs that are induced by their signals (...) here, agents do not form beliefs about any particular other agents. They only form beliefs about the crosssection distribution of the other agents' characteristics.*"

endowment to consume or to give to the pool of contributions to the public good.

In this setup, we prove that there is always a symmetric equilibrium; that is, a Nash equilibrium where all players with the same characteristic (same type and belief) make identical contributions. In such symmetric equilibrium, only players with the highest preference type and the most pessimistic belief (i.e., the one with the lowest proportion of highest type players) make positive contributions, provided that there is at least one such player. In this situation, the total contribution for the public good could be greater, smaller or equal to the total contribution under an analogous game of complete information and heterogeneous preferences. It is greater if and only if the players that actually contribute are pessimistic in absolute terms, which means that they believe that the number of highest type players is smaller than it actually is. This result allows us to evaluate how the classical free rider problem depends on the heterogeneity of beliefs and on the preferences present in the population. If there are many possible types and players, and only few of them have the most pessimistic belief, then the free rider problem is severe. This is because only a relatively small minority of players contributes and these contributions are relatively large. Moreover, contributors may be better off than non-contributors of different types. This is true because contributors benefit more from the same amount of public good than players with a different preference type.

If there is no individual who simultaneously has the highest preference type and the most pessimistic belief, a tradeoff between preferences and beliefs arises. Depending on the parameters, there may be symmetric equilibria where all contributions are made by players with the same characteristic or where players with different characteristics share the contributions. There may also exist multiple symmetric equilibria, including cases where some players with the highest preference type do not contribute, while, at the same time, players with the second highest preference type contribute. When there are exactly two possible preference types, the symmetric equilibrium is unique again. In this equilibrium, high type players with the second most pessimistic beliefs in the population may be the only contributors, or they may share the provision of the public good with the most pessimistic players of low type. These two cases illustrate the tension between

preferences and beliefs. The total contribution may be greater, smaller or equal to the total contribution under an analogous game of complete information. It is greater (smaller) if and only if the second most pessimistic belief is pessimistic (optimistic) in absolute terms.

The next section describes the notation and the basic setup of the model. Section 3 calculates the symmetric equilibrium and describes its properties. The only contributors are those players with the greatest public good value and with the most pessimistic belief. Section 4 analyzes the game when these players are absent. Section 5 concludes the paper and the Appendix contains all proofs.

2 Notation and Setup

There is a finite set J with exactly $N = |J| \in \{2, 3, \dots\}$ players (potential contributors for the public good). Depending on their preferences for public good consumption, each player has a type $t \in T = \{1, 2, \dots, |T|\}$.

Individuals may also differ in their beliefs about the distribution of types in the population. Let I denote the set of possible beliefs; that is, possible probability distributions over the set of types. There are exactly $|I| \in \{2, 3, \dots\}$ different beliefs. For each $i \in I$, let $\delta^{(i)} = (\delta_1^{(i)}, \dots, \delta_{|T|}^{(i)})$, where $\delta_t^{(i)}$ represents the probability of type $t \in T$ according to any player having belief $\delta^{(i)}$. As usual, $\delta_1^{(i)} + \dots + \delta_{|T|}^{(i)} = 1$, for every $i \in I$, and we suppose that $\delta_t^{(i)} > 0$, for every pair (t, i) .

The characteristic of a player is a pair $(t, i) \in T \times I$. Let $J(t, i)$ be the set of all players with characteristic (t, i) , for each $t \in T$ and $i \in I$ and suppose that $J(t, i) \neq \emptyset$.³ Let $N_t^{(i)} = |J(t, i)|$ be the number of players with characteristic (t, i) . For each $t \in T$, let $N_t = \sum_{i \in I} N_t^{(i)}$ be the number of type t individuals. Hence, $N = \sum_{t \in T} N_t$ is the total number of players. The share of individuals of type t with belief $\delta^{(i)}$ is $\beta_t^{(i)} = N_t^{(i)} / N_t$. Thus, $\sum_{i \in I} \beta_t^{(i)} = 1$. There is perfect information regarding the *conditional* distributions of beliefs given the types. For each type $t \in T$, all players know the proportions of type t players that have each belief.

³Section 4 assumes that $J(1, 1)$ is an empty set in order to find and analyze the relationship between beliefs and preferences that determines the symmetric equilibrium.

When G units of the public good are provided, a player j with characteristic (t, i) consuming x_j units of numéraire obtains utility $U_j(x_j, G) = x_j + \lambda_t \ln(G)$, where the constants λ_t , with $t \in T$, are ordered by: $\lambda_1 > \lambda_2 > \dots > \lambda_{|T|} > 0$.

Assume that the values $\delta_1^{(1)}, \delta_1^{(2)}, \dots, \delta_1^{(|I|)}$ never coincide and $\delta_1^{(1)} = \min\{\delta_1^{(i)} \mid i = 1, \dots, |I|\}$. These are the proportions of players with type $t = 1$, according to the $|I|$ different beliefs in I . Thus, belief $\delta^{(1)}$ is the one of players assigning the minimum for the proportion of players of type $t = 1$ in the population. Because type $t = 1$ players have the maximal propensity to contribute to the public good ($\lambda_1 = \max\{\lambda_t \mid t \in T\}$), we refer to individuals with belief $\delta^{(1)}$ as the most pessimistic ones; that is, distribution $\delta^{(1)}$ is the most pessimistic in the sense that it has the lowest proportion of players of type $t = 1$.

For simplicity, suppose that all players have the same initial endowment $w > 0$. Each individual $j \in J(t, i)$ chooses her contribution g_j in order to:

$$\text{Max}_{g_j \geq 0} x_j + \lambda_t \ln G_{t,i}(g_j), \quad (1)$$

subject to:

$$x_j + g_j = w,$$

where $G_{t,i}(g_j)$ is the total contribution perceived by $j \in J(t, i)$. We will specify that function in the next subsection. Assume that the common initial endowment w is sufficiently large, avoiding positive corner solutions.

Define the simultaneous moves game of public good provision with individuals having heterogeneous beliefs regarding the population distribution of types by:

$$\mathbb{P} = \left\{ J, T, I, \{\beta_t^{(i)}, \delta_t^{(i)}\}_{t \in T; i \in I}, \{\lambda_t\}_{t \in T} \right\}.$$

Remark 1 *Throughout this paper, we assume a logarithmic quasi-linear specification for the utility function of individuals. Quasi-linearity is a restrictive assumption; however, the same qualitative results may be obtained if instead of $\ln(G)$ we use any function $\phi(G)$ with $\phi' > 0$, $\phi'' < 0$, $\phi'(0) = +\infty$, and $\phi'(+\infty) = 0$. Indeed, the first order condition $G_{i,t} \geq \lambda_t$ (to be stated in (2)) becomes $G_{i,t} \geq \tilde{\lambda}_t$, where $\tilde{\lambda}_t = (\phi')^{-1}(1/\lambda_t)$. The existence of $(\phi')^{-1}(1/\lambda_t)$ is guaranteed by the monotonicity of ϕ' , $\phi'(0) = +\infty$, and $\phi'(+\infty) = 0$.*

3 Symmetric Equilibrium

The lack of determination of individuals in the same set $J(t, i)$ leads us to consider the symmetric equilibrium concept; which is a Nash equilibria where all players in $J(t, i)$ make the same contribution, denoted $g_t^{(i)}$.⁴ Therefore, a symmetric equilibrium for the game \mathbb{P} is a profile of contributions $(g_t^{(i)})_{i \in I, t \in T}$ such that, for each player $j \in J(t, i)$, her contribution is $g_j = g_t^{(i)}$, a solution of problem (1), given that the contributions of all other players are those prescribed in the symmetric equilibrium. In other words, in the symmetric equilibrium, no player has a unilateral profitable deviation, and every pair of players with the same characteristic make identical contributions.

Let $(t, i) \in T \times I$. Fix a player $j \in J(t, i)$. She believes that there are $\delta_\tau^{(i)} \beta_\tau^{(k)} N$ individuals in $J(\tau, k)$, for each $\tau \in T$ and $k \in I$, each one of them contributing $g_\tau^{(k)}$ in a symmetric equilibrium. If j decides that her contribution is g_j , the total amount of public good provision perceived by her becomes:

$$G_{t,i}(g_j) = g_j - g_t^{(i)} + \sum_{k \in I} \sum_{\tau \in T} \beta_\tau^{(k)} \delta_\tau^{(i)} N g_\tau^{(k)}.$$

Solving player j 's maximization problem (1), the second order condition always holds. Hence, the solution is given by her first order condition:

$$G_{t,i}(g_j^*) \geq \lambda_t, \quad \text{with "=" if } g_j^* > 0.$$

Equivalently:

$$g_j^* - g_t^{(i)} + \sum_{k \in I} \sum_{\tau \in T} \beta_\tau^{(k)} \delta_\tau^{(i)} N g_\tau^{(k)} \geq \lambda_t, \quad \text{with "=" if } g_j > 0. \quad (2)$$

The next result establishes the existence and uniqueness of the symmetric equilibrium.

Proposition 1 *Suppose that the subset of players $J(1, 1)$ is non-empty. Then, there exists a unique symmetric equilibrium $(g_t^{(i)})_{i \in I, t \in T}$ for the game \mathbb{P} . The set of all symmetric*

⁴There are many other Nash equilibria. In fact, if two players are making positive contributions in a Nash equilibrium, then there is another Nash equilibrium where all other players play as before and the two players slightly redistribute their contributions; the amount that one increases equals the decrease in the other. This works as long as their final contributions are still non-negative.

equilibria is characterized by the following conditions, one for each pair $(t, i) \in T \times I$:

$$\sum_{k \in I} \sum_{\tau \in T} \beta_{\tau}^{(k)} \delta_{\tau}^{(i)} g_{\tau}^{(k)} \geq \frac{\lambda_t}{N}, \quad \text{with "=" if } g_t^{(i)} > 0. \quad (3)$$

In the unique symmetric equilibrium, contributions of all players are zero, except for those with characteristic $(t, i) = (1, 1)$. The only players making positive contributions are the most pessimistic (belief $\delta^{(1)}$) of those who benefit most from the public good (type $t = 1$).⁵

Formally, contributions are:

$$g_1^{(1)} = \frac{\lambda_1}{\delta_1^{(1)} \beta_1^{(1)} N}, \quad \text{and} \quad (4)$$

$$g_t^{(i)} = 0, \quad \text{if } (t, i) \neq (1, 1). \quad (5)$$

Andreoni [1] proves that if there is incomplete information about individual wealth, then the proportion of contributors decreases as the number of players increases. By contrast, Proposition 1 reveals that the proportion of contributors may increase or decrease depending on the proportion of players with characteristic $(1, 1)$ among the entrants.

Consider two games with identical parameters, except for beliefs. These games have the same symmetric equilibrium if and only if the belief parameter $\delta_1^{(1)}$ coincides. This neutrality result is similar to that of Gradstein et al. [6] who proved that the total amount of the public good is invariant to wealth redistributions.

Let $\delta^* = (\delta_1^*, \dots, \delta_{|T|}^*)$ be the true distribution of types; namely $\delta_t^* = N_t/N$. Let G^* denote the aggregate provision of the public good in the symmetric equilibrium. Then:

$$G^* = \sum_{i \in I} \sum_{t \in T} \beta_t^{(i)} \delta_t^* g_t^{(i)} N = \beta_1^{(1)} \delta_1^* g_1^{(1)} N.$$

Substituting equation (4) into this expression:

$$G^* = \frac{\delta_1^* \lambda_1}{\delta_1^{(1)}}. \quad (6)$$

When all individuals have correct beliefs about the distribution of preferences for public good in the population, then $G^* = \lambda_1$. If the most pessimistic belief in relative

⁵If we change the hypothesis $\delta_1^{(1)} < \delta_1^{(2)}$ to $\delta_1^{(1)} = \delta_1^{(2)}$, then players with belief $\delta^{(2)}$ and type $t = 1$ will also make positive contributions to the public good.

terms, namely $\delta_1^{(1)}$, is also pessimistic in absolute terms; that is $\delta_1^{(1)} < \delta_1^*$, then the aggregate contribution is larger than what it would be if all players had correct beliefs about the preferences' distribution. Thus, $\delta_1^{(1)} < \delta_1^*$ if and only if $G^* > \lambda_1$.

If the subset $J(1, 1)$ is non-empty, players with characteristic $(1, 1)$ know this. In particular, they know that $\delta_1^{(1)} \geq 1/N$. This inequality establishes the following upper bound for G^* :

$$G^* \leq N\delta_1^*\lambda_1 = N_1\lambda_1.$$

If $j \in J(\tau, i)$, let $t(j) = \tau$. In order to calculate the efficient provision of the public good, a central planner chooses individual contributions $\{z_j\}_{j \in J}$ in order to solve the problem:

$$\text{Max}_{\{z_j\}_{j \in J}} \sum_{j \in J} (w - z_j + \lambda_{t(j)} \ln(G)) \quad (7)$$

subject to:

$$z_j \geq 0, \text{ for every } j \in J,$$

and:

$$G = \sum_{j \in J} z_j.$$

There are multiple solutions $\{z_j\}_{j \in J}$ for the problem (7). However, all solutions lead to the same total contribution:

$$G^e = \sum_{j \in J} \lambda_{t(j)} = N \sum_{t \in T} \delta_t^* \lambda_t = \sum_{t \in T} N_t \lambda_t. \quad (8)$$

Corollary 1 *Suppose that the subset of players $J(1, 1)$ is non-empty. The total contribution in the symmetric equilibrium of the game \mathbb{P} is $G^* = (\delta_1^*/\delta_1^{(1)})\lambda_1$. The greater the pessimism of contributors (lower $\delta_1^{(1)}$), the greater the total contribution in the symmetric equilibrium. In addition, the equilibrium contribution G^* is lower than the efficient contribution G^e given by equation (8).*

There are some options for mitigating the classic under-provision of public good (free rider problem). Bag and Roy [3] show that the total amount of public good increases if the

game is played in two stages, with first round contributions announced before the second round. Bag and Roy [4] prove that the total provision of public good is larger in a two period game where players may contribute only once, than in the static game. Corollary 1 states that, in our static game, the total contribution increases when the contributors become more pessimistic.

Despite bearing the burden of providing the public good, the most pessimistic players may attain greater payoffs than some of the defectors. This is the content of the next result.

Proposition 2 *Let $u_{t,i} = U_j(w - g_t^{(i)}, G^*)$, for each $(t, i) \in T \times I$ and each player $j \in J(t, i)$. Then:*

(i) *Among individuals of type $t = 1$, the most pessimistic players (i.e., contributors) are always worse off than other players with the same preferences. Formally, $u_{1,1} < u_{1,i}$, for all $i = 2, 3, \dots, |I|$.*

(ii) *Contributors may be better off or worse off than players with different types. For each $i \in I$ and $t \in T - \{1\}$:*

$$u_{1,1} > u_{t,i} \quad \Leftrightarrow \quad N > \frac{\lambda_1}{(\lambda_1 - \lambda_t) \ln(G^*) \delta_1^{(1)} \beta_1^{(1)}}.$$

As N increases, players with characteristic $(1, 1)$ become happier because G^* remains constant and the individual contribution $g_1^{(1)}$ decreases, according to equation (4). The utilities of all individuals with characteristic (t, i) , for $i \in I$ and $t \in T - \{1\}$, do not depend on N . Hence, as N grows, eventually, the utility of contributors becomes larger than the utilities of defectors of types $t \in T - \{1\}$.

4 Beliefs versus Preferences

This section examines the game in absence of players with characteristic $(t, i) = (1, 1)$; that is, the only individuals who were making contributions for the public good in the previous analysis. This new analysis sheds light on the tradeoff between beliefs and preferences in the provision of public good.

Consider the beliefs ordered by their first component (the probability of being of type $t = 1$), and order the beliefs in such a way that $\delta_1^{(1)} < \delta_1^{(2)} < \dots < \delta_1^{(I)}$. Suppose that there is no player with characteristic $(1, 1)$; that is, $J(1, 1) = \emptyset$. To calculate the new symmetric equilibria we use system (3) for all $(t, i) \neq (1, 1)$.

Proposition 3 *If there is no player with characteristic $(t, i) = (1, 1)$, then there exist at least one symmetric equilibrium. Every symmetric equilibrium $(g_t^{(i)})_{(t,i) \in T \times I - \{(1,1)\}}$ satisfies $g_t^{(i)} = 0$, for all $i \geq 2, t \geq 2$; and also $g_t^{(1)} = 0$, for all $t \geq 3$.*

Proposition 3 asserts that in every symmetric equilibrium, only the individuals who place the greatest value on the public good, or the most pessimistic individuals who place the second largest value on the public good, may contribute. The next example shows some symmetric equilibria that may arise.

Example 1 *Consider the following specification: $N = 100, T = \{1, 2, 3\}, I = \{1, 2, 3\}, \lambda_1 = 200, \lambda_2 = 100, 0 < \lambda_3 < 100$, and the beliefs $\delta^{(1)}, \delta^{(2)}, \delta^{(3)}$ are the rows of the following matrix:*

$$\delta = \begin{bmatrix} 0.33 & 0.1 & 0.57 \\ 0.5 & 0.4 & 0.1 \\ 0.75 & 0.15 & 0.1 \end{bmatrix}.$$

Let $\bar{g}_1 = \beta_1^{(2)} g_1^{(2)} + \beta_1^{(3)} g_1^{(3)}$ and $\bar{g}_2 = \beta_2^{(1)} g_2^{(1)}$. The first order conditions characterizing the equilibria are (see proof of Proposition (3)):

$$\begin{aligned} \delta_1^{(1)} \bar{g}_1 + \delta_2^{(1)} \bar{g}_2 &= 0.33 \bar{g}_1 + 0.1 \bar{g}_2 \geq \frac{\lambda_2}{N} = 1; & \text{with "=" if } g_2^{(1)} > 0, \\ \delta_1^{(2)} \bar{g}_1 + \delta_2^{(2)} \bar{g}_2 &= 0.5 \bar{g}_1 + 0.4 \bar{g}_2 \geq \frac{\lambda_1}{N} = 2; & \text{with "=" if } g_1^{(2)} > 0, \\ \delta_1^{(3)} \bar{g}_1 + \delta_2^{(3)} \bar{g}_2 &= 0.75 \bar{g}_1 + 0.15 \bar{g}_2 \geq \frac{\lambda_1}{N} = 2; & \text{with "=" if } g_1^{(3)} > 0. \end{aligned}$$

Figure 1 shows the position of each line above for the given parameter values.

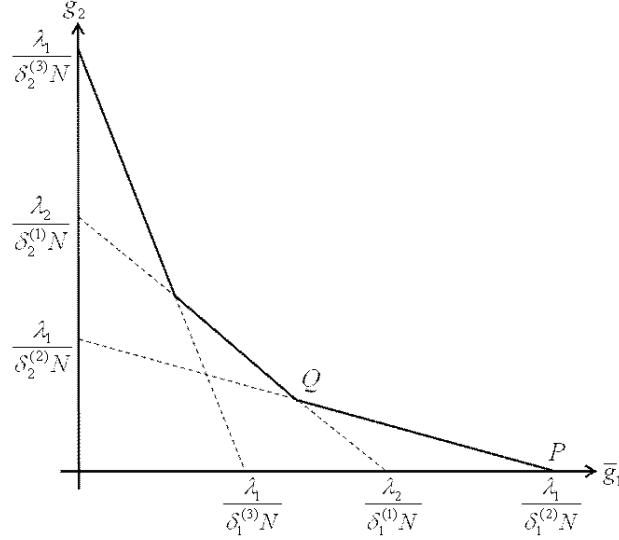


Figure 1: Multiplicity of equilibria in the public good contribution model with heterogeneous beliefs

The point P corresponds to the solution $g_1^{(2)} = 4/\beta_1^{(2)}$, $g_1^{(3)} = 0$ (or $\bar{g}_1 = 4$) and $g_2^{(1)} = 0$ (or $\bar{g}_2 = 0$). In addition, the point Q corresponds to another equilibrium: $\bar{g}_1 = 2.4$ and $\bar{g}_2 = 2$, and then, $g_1^{(2)} = 2.4/\beta_1^{(2)}$, $g_1^{(3)} = 0$ and $g_2^{(1)} = 2/\beta_2^{(1)}$.

If the parameter values are not those in matrix δ , but instead they are such that point Q is the intersection of the three lines in Figure 1, then there is a symmetric equilibrium with positive contributions $g_1^{(2)} > 0$, $g_1^{(3)} > 0$ and $g_2^{(1)} > 0$.

If there are only two possible types for the preferences ($|T| = 2$), multiple symmetric equilibria cannot occur. The following result characterizes the symmetric equilibrium in terms of the parameters.

Proposition 4 Suppose that $T = 2$ and there is no player with characteristic $(1, 1)$. Then, the unique symmetric equilibrium $(g_t^{(i)})_{(t,i) \in T \times I - \{(1,1)\}}$ satisfies $g_t^{(i)} = 0$, for all $(t, i) \notin \{(1, 2), (2, 1)\}$, and:

(i) If $\frac{\lambda_2}{\lambda_1} < \frac{\delta_1^{(1)}}{\delta_1^{(2)}}$, then $g_1^{(2)} = \frac{\lambda_1}{\beta_1^{(2)} \delta_1^{(2)} N} > 0$ and $g_2^{(1)} = 0$.

(ii) If $\frac{\lambda_2}{\lambda_1} \geq \frac{\delta_1^{(1)}}{\delta_1^{(2)}}$, then $g_1^{(2)} = \frac{\delta_2^{(1)} \lambda_1 - \delta_2^{(2)} \lambda_2}{\beta_1^{(2)} (\delta_1^{(2)} - \delta_1^{(1)}) N} > 0$ and $g_2^{(1)} = \frac{\delta_1^{(2)} \lambda_2 - \delta_1^{(1)} \lambda_1}{\beta_2^{(1)} (\delta_1^{(2)} - \delta_1^{(1)}) N} \geq 0$.

(iii) In both cases, the aggregate contribution of players with characteristic $(t, i) = (1, 2)$ is larger than the aggregate contribution of players with characteristic $(t, i) = (2, 1)$. Formally, $\beta_1^{(2)} g_1^{(2)} > \beta_2^{(1)} g_2^{(1)}$.

Independently of beliefs, the players who most value the public good contribute to its provision. Players who value the public good the second highest *may* contribute, depending on the parameter values. The value λ_2/λ_1 is the ratio of preference parameters between "low" types ($t = 2$) and "high" types ($t = 1$). The greater (lower) this parameter, the greater (lower) the public good value for individuals of type $t = 2$. The ratio $\delta_1^{(1)}/\delta_1^{(2)}$ measures how optimistic the belief $\delta^{(1)}$ is when compared to belief $\delta^{(2)}$. The greater (lower) this ratio, the more (less) optimistic the belief $\delta^{(1)}$.

Part (i) of Proposition 4 establishes that if the ratio of preference parameters is low when compared to the relative optimism of $\delta^{(1)}$, then only players of type $t = 1$ and belief $\delta^{(2)}$ contribute. Intuitively, this indicates that preferences dominate beliefs; only those who gain the most from a given amount of public good contribute, even though there are more pessimistic non-contributing players.

Part (ii) asserts that if the ratio of preference parameters is sufficiently high compared to the relative optimism of belief $\delta^{(1)}$, then individuals with characteristics $(t, i) = (1, 2)$ and $(t, i) = (2, 1)$ contribute. This is how preferences and beliefs interact.

The next result compares the total contribution to the public good provision in this setting with the aggregate contribution in the framework of the previous section. Let G^{**} denote the aggregate contribution in the equilibrium of Proposition 4.

Corollary 2 *If $T = 2$ and there is no player with characteristic $(t, i) = (1, 1)$, then, in the symmetric equilibrium described by Proposition 4:*

(i) *If $\frac{\lambda_2}{\lambda_1} < \frac{\delta_1^{(1)}}{\delta_1^{(2)}}$, then:*

$$G^{**} = \frac{\delta_1^*}{\delta_1^{(2)}} \lambda_1 < \frac{\delta_1^*}{\delta_1^{(1)}} \lambda_1 = G^*.$$

(ii) If $\frac{\lambda_2}{\lambda_1} \geq \frac{\delta_1^{(1)}}{\delta_1^{(2)}}$, then:

$$G^{**} = \left(\frac{\delta_1^* - \delta_1^{(1)}}{\delta_1^{(2)} - \delta_1^{(1)}} \right) \lambda_1 + \left(\frac{\delta_1^{(2)} - \delta_1^*}{\delta_1^{(2)} - \delta_1^{(1)}} \right) \lambda_2. \quad (9)$$

In particular, if $\delta_1^{(1)} < \delta_1^*$, then:

$$G^{**} \leq \frac{\delta_1^*}{\delta_1^{(1)}} \lambda_2 < \frac{\delta_1^*}{\delta_1^{(1)}} \lambda_1 = G^*.$$

(iii) In both cases above:

$$G^{**} > \lambda_1 \quad \Leftrightarrow \quad \delta_1^{(2)} < \delta_1^*.$$

If the inequality defining Part (ii) is strict; that is, $\lambda_2/\lambda_1 > \delta_1^{(1)}/\delta_1^{(2)}$, then it is possible that $G^{**} > G^*$. This occurs when δ_1^* is sufficiently close to zero. Indeed, as $\delta_1^* \rightarrow 0$, then $G^{**} \rightarrow (-\delta_1^{(1)}\lambda_1 + \delta_1^{(2)}\lambda_2)/(\delta_1^{(2)} - \delta_1^{(1)})$ and $G^* \rightarrow 0$; thus, the inequalities $G^{**} > G^*$ and $\lambda_2/\lambda_1 > \delta_1^{(1)}/\delta_1^{(2)}$ become equivalent.⁶

Part (iii) of Corollary 2 establishes that the aggregate contribution in equilibrium is greater than it would be in the complete information game exactly when the belief $\delta^{(2)}$ is pessimistic in absolute terms.

Regardless of the case, if $\delta_1^{(1)} \leq \delta_1^* \leq \delta_1^{(2)}$, then the total provision for the public good is a convex combination of λ_1 and λ_2 . Indeed, by Part (iii), the hypothesis $\delta_1^* \leq \delta_1^{(2)}$ implies that $G^{**} \leq \lambda_1$. If $\lambda_2/\lambda_1 < \delta_1^{(1)}/\delta_1^{(2)}$, then $G^{**} = \delta_1^*\lambda_1/\delta_1^{(2)} \geq \delta_1^{(1)}\lambda_1/\delta_1^{(2)} > \lambda_2$. If $\lambda_2/\lambda_1 \geq \delta_1^{(1)}/\delta_1^{(2)}$, then formula (9) establishes that G^{**} is a convex combination of λ_1 and λ_2 because the coefficients of λ_1 and λ_2 must be positive given the hypotheses $\delta_1^{(1)} \leq \delta_1^* \leq \delta_1^{(2)}$ and $\delta_1^{(1)} < \delta_1^{(2)}$.

5 Conclusions

This work analyzes a game of public good contribution in a population of anonymous individuals who are heterogeneous in their preferences for the public good and also in their

⁶This argument is based on the continuity of the functions $\delta_1^* \mapsto G^*$ and $\delta_1^* \mapsto G^{**}$.

beliefs about the population distribution over those preferences. In line with the literature, only players who most value the public good bear the burden of the contribution. However, not all of those individuals contribute in the unique symmetric equilibrium; only the most pessimistic ones. If that set of individuals is empty, there is a tension between beliefs and preferences and multiple symmetric equilibria may arise. When there are only two preference types, uniqueness of symmetric equilibrium emerges. In such a case, depending on the dominance of preferences over beliefs, there may be contributions only of high type individuals with the second most pessimistic beliefs in the population, or a share of the total contribution between that group and the most pessimistic players of low preference type.

The public good is under-provided, as usual in similar games. When the set of players who most value the public good and who hold the most pessimistic belief is non-empty, the aggregate contribution is greater than the total contribution under complete information if and only if the most pessimistic belief, denoted $\delta^{(1)}$, is pessimistic in absolute terms. Absolute pessimism means that the belief $\delta^{(1)}$ assigns smaller probability to the highest preference type than the true probability. If that subset of players is empty and there are only two preference types, the aggregate contribution is greater than the total contribution under complete information if and only if the second most pessimistic belief in the population, denoted $\delta^{(2)}$, is pessimistic in absolute terms. These results establish a remarkable conclusion: if the policy maker does not take into account the existence of beliefs heterogeneity, he may expect a total contribution significantly different from the actual one, in either direction.

Further research could analyze a similar game in a dynamic context or when the preferences are not quasi-linear. The first extension will incorporate mechanisms that reveal (at least in part) the incomplete information. The second proposal could shed light on the importance of wealth effects.

6 Appendix

This appendix contains all proofs.

Proof. (Proposition 1) Let $(g_t^{(i)})_{(t,i) \in T \times I}$ be a symmetric equilibrium. Fix $(t, i) \in T \times I$. If $j \in J(t, i)$, then $g_j^* = g_t^{(i)}$. Substituting this value in the first order condition (2) we obtain (3). Reciprocally, suppose that $(g_t^{(i)})_{(t,i) \in T \times I}$ satisfies condition (3) and all players, except j , play according to their characteristics and the given profile. When $g_j^* = g_t^{(i)}$, condition (2) holds because $(g_t^{(i)})_{(t,i) \in T \times I}$ satisfies (3). Therefore, $(g_t^{(i)})_{(t,i) \in T \times I}$ is a Nash equilibrium.

It is straightforward to check that the profile given by equations (4) and (5) satisfies condition (3).

Finally, let us prove the uniqueness of the symmetric equilibrium. For each $\tau \in T$, the type-conditional expected contribution is:

$$\bar{g}_\tau = \sum_{k \in I} \beta_\tau^{(k)} g_\tau^{(k)}. \quad (10)$$

The system (3) is composed of $|I| \times |T|$ equations that can be solved as follows. Fix $i \in I$ and consider the following $|T|$ equations sub-system:

$$\sum_{\tau \in T} \delta_\tau^{(i)} \bar{g}_\tau \geq \frac{\lambda_t}{N}; \quad \text{with "=" if } g_t^{(i)} > 0, \quad t = 1, \dots, |T|.$$

The first term in the inequalities above is the same for all t . Thus, $g_t^{(i)} = 0$, for every $t = 2, \dots, |T|$, because the λ 's are ordered ($\lambda_1 > \dots > \lambda_{|T|}$). Therefore, $\bar{g}_\tau = 0$ for all $\tau \geq 2$ and $\sum_{\tau \in T} \delta_\tau^{(i)} \bar{g}_\tau = \delta_1^{(i)} \bar{g}_1$. Finally, consider the first equation in each group given above. Then, there are $|I|$ conditions, namely:

$$\sum_{\tau \in T} \delta_\tau^{(i)} \bar{g}_\tau = \delta_1^{(i)} \bar{g}_1 \geq \frac{\lambda_1}{N}; \quad \text{with "=" if } g_1^{(i)} > 0, \quad i = 1, \dots, |I|.$$

For any equilibrium, it is necessary that $\bar{g}_1 \neq 0$ (otherwise, the system above would imply $\delta_1^{(i)} \bar{g}_1 = 0 \geq \lambda_1/N$, but by assumption $\lambda_1 > 0$ and $N > 0$). Using that $\delta_1^{(1)} = \min\{\delta_1^{(i)} \mid i = 1, \dots, |I|\}$, we conclude that $g_1^{(i)} = 0, i = 2, \dots, |I|$, and $\bar{g}_1 = \beta_1^{(1)} g_1^{(1)}$. Equalizing this to λ_1/N , we obtain the solution (4). ■

Proof. (Corollary 1) Because $1/\delta_1^{(1)} \leq N$ and $N\delta_1^* = N_1$, then $G^* = \lambda_1 \delta_1^* / \delta_1^{(1)} \leq N\delta_1^* \lambda_1 = N_1 \lambda_1 < \sum_{t \in T} N_t \lambda_t = G^e$. ■

Proof. (Proposition 2) Part (i). This results from the fact that $u_{1,1} = w - g_1^{(1)} + \lambda_1 \ln(G^*)$, and $u_{1,i} = w + \lambda_1 \ln(G^*)$, for every $i = 2, \dots, |I|$.

Part (ii). For every $i \in I$ and $t \geq 2$, then $u_{t,i} = w + \lambda_t \ln(G^*)$. Hence, $u_{1,1} > u_{t,i}$ if and only if $g_1^{(1)} < (\lambda_1 - \lambda_t) \ln(G^*)$. Using the equilibrium value of $g_1^{(1)}$ given by equation (4), after some simple algebra, we obtain this result. ■

Proof. (Proposition 3) The argument for existence is similar to the one in the proof of Proposition 1. When there is no player with characteristic (1, 1), the equilibrium is defined by the solution of the following system: for each $(t, i) \in C \equiv T \times I - \{(1, 1)\}$:

$$\sum_{(\tau, k) \in C} \delta_\tau^{(i)} \beta_\tau^{(k)} g_\tau^{(k)} \geq \frac{\lambda_t}{N}; \quad \text{with "=" if } g_t^{(i)} > 0. \quad (11)$$

Fix $i \geq 2$. Consider the following sub-system, with $|T|$ equations, one for each $t = 1, \dots, |T|$:

$$\sum_{(\tau, k) \in C} \delta_\tau^{(i)} \beta_\tau^{(k)} g_\tau^{(k)} \geq \frac{\lambda_t}{N}; \quad \text{with "=" if } g_t^{(i)} > 0.$$

The terms in the left-hand side of the inequalities above is the same, regardless of the value of t . Hence, $g_2^{(i)} = g_3^{(i)} = \dots = g_{|T|}^{(i)} = 0$ because $\lambda_1 > \lambda_2 > \dots > \lambda_{|T|}$. Indeed, if $g_t^{(i)} > 0$, for some $t > 1$, then the corresponding condition would hold with "=". Thus, $\lambda_t/N = \sum_{(\tau, k) \in C} \delta_\tau^{(i)} \beta_\tau^{(k)} g_\tau^{(k)} \geq \lambda_1/N$, contradicting $\lambda_1 > \lambda_t$.

For $i = 1$, there is only a sub-system with $|T| - 1$ equations. For each $t = 2, \dots, |T|$:

$$\sum_{(\tau, k) \in C} \delta_\tau^{(1)} \beta_\tau^{(k)} g_\tau^{(k)} \geq \frac{\lambda_t}{N}; \quad \text{with "=" if } g_t^{(1)} > 0.$$

Once again, the first term in the inequalities above is the same for all t . Thus, $g_3^{(1)} = g_4^{(1)} = \dots = g_{|T|}^{(1)} = 0$ because $\lambda_2 > \lambda_3 > \dots > \lambda_{|T|}$.

Using $g_t^{(i)} = 0$, for all $t \geq 2$ and $i \geq 2$, and also $g_t^{(1)} = 0$, for all $t \geq 3$, then $\bar{g}_1 = \beta_1^{(2)} g_1^{(2)} + \dots + \beta_1^{(|I|)} g_1^{(|I|)}$, $\bar{g}_2 = \beta_2^{(1)} g_2^{(1)}$, and:

$$\sum_{(\tau, k) \in C} \delta_\tau^{(i)} \beta_\tau^{(k)} g_\tau^{(k)} = \delta_1^{(i)} \left(\beta_1^{(2)} g_1^{(2)} + \dots + \beta_1^{(|I|)} g_1^{(|I|)} \right) + \delta_2^{(i)} \beta_2^{(1)} g_2^{(1)} = \delta_1^{(i)} \bar{g}_1 + \delta_2^{(i)} \bar{g}_2.$$

By the first order conditions (11), for all $(t, i) \in \{(2, 1), (1, 2), (1, 3), \dots, (1, |I|)\}$:

$$\delta_1^{(i)} \bar{g}_1 + \delta_2^{(i)} \bar{g}_2 \geq \frac{\lambda_t}{N}; \quad \text{with "=" if } g_t^{(i)} > 0.$$

Hence, the equations defining the equilibrium are:

$$\delta_1^{(1)} \bar{g}_1 + \delta_2^{(1)} \bar{g}_2 \geq \frac{\lambda_2}{N}; \quad \text{with "=" if } g_2^{(1)} > 0, \quad (12)$$

and, for all $i = 2, \dots, |I|$:

$$\delta_1^{(i)} \bar{g}_1 + \delta_2^{(i)} \bar{g}_2 \geq \frac{\lambda_1}{N}; \quad \text{with "=" if } g_1^{(i)} > 0. \quad (13)$$

In order to prove that the system (12) and (13) has a solution, consider two cases.

Case I: Suppose that $\lambda_2/\lambda_1 \leq \delta_1^{(1)}/\delta_1^{(2)}$.

The contributions $g_2^{(1)} = 0$, $g_1^{(2)} = \lambda_1/(\delta_1^{(2)}\beta_1^{(2)}N)$, and $g_1^{(k)} = 0$, for every $k \geq 3$, define a solution for the system (12) and (13). Indeed, with those contributions, $\bar{g}_1 = \lambda_1/(\delta_1^{(2)}N)$ and $\bar{g}_2 = 0$ will satisfy conditions (12) and (13), given the parameter conditions of this case and $\delta_1^{(1)} < \delta_1^{(2)} < \dots < \delta_1^{(|I|)}$.

Case II:⁷ Suppose that $\lambda_2/\lambda_1 > \delta_1^{(1)}/\delta_1^{(2)}$.

Consider two sub-cases:

Sub-case IIa) $\lambda_2/\lambda_1 \geq \delta_2^{(1)}/\delta_2^{(k)}$, for every $k \geq 2$. We claim that $g_2^{(1)} = \lambda_2/(\delta_2^{(1)}\beta_2^{(1)}N)$ and $g_1^{(k)} = 0$, for all $k \geq 2$ is a symmetric equilibrium. Indeed, in that contributions profile, $\bar{g}_1 = 0$ and $\bar{g}_2 = \lambda_2/(\delta_2^{(1)}N)$. This satisfies condition (12) with "=" and condition (13) with " \geq " for all $i \geq 2$.

Sub-case IIb) $\exists k \geq 2$ such that $\lambda_2/\lambda_1 < \delta_2^{(1)}/\delta_2^{(k)}$. Define the set of indices $K = \left\{ k \geq 2 \mid \lambda_2/\lambda_1 < \delta_2^{(1)}/\delta_2^{(k)} \right\}$. Consider, for each $m \in K$, the linear system:

$$\begin{aligned} \delta_1^{(1)} \bar{g}_1 + \delta_2^{(1)} \bar{g}_2 &= \frac{\lambda_2}{N} \\ \delta_1^{(m)} \bar{g}_1 + \delta_2^{(m)} \bar{g}_2 &= \frac{\lambda_1}{N}; \end{aligned} \quad (14)$$

We claim that, for each $m \in K$, the corresponding system has a strictly positive

⁷The arguments in this proof follow from geometric constructions in the plane $\bar{g}_1 \times \bar{g}_2$. The idea is to fix the straight segments defined by condition (13) and to consider two possibilities for the position of the segment given by condition (12). In sub-case (IIa), the line (12) is above all segments defined by condition (13). In sub-case (IIb), the line (12) intersects at least one of the segments (13).

solution $\bar{g}_1 > 0$ and $\bar{g}_2 > 0$. Let $\Delta = \delta_1^{(1)}\delta_2^{(m)} - \delta_2^{(1)}\delta_1^{(m)}$. The solution of system (14) is:

$$\begin{aligned}\bar{g}_1 &= \frac{\delta_2^{(m)}\lambda_1}{N\Delta} \left[\frac{\lambda_2}{\lambda_1} - \frac{\delta_2^{(1)}}{\delta_2^{(m)}} \right], \\ \bar{g}_2 &= \frac{\delta_1^{(m)}\lambda_1}{N\Delta} \left[-\frac{\lambda_2}{\lambda_1} + \frac{\delta_1^{(1)}}{\delta_1^{(m)}} \right],\end{aligned}\tag{15}$$

From the hypothesis $\lambda_2/\lambda_1 > \delta_1^{(1)}/\delta_1^{(2)}$ (Case II) and because $m \in K$, then $\delta_1^{(1)}/\delta_1^{(2)} < \lambda_2/\lambda_1 < \delta_2^{(1)}/\delta_2^{(m)}$, which implies that $\delta_1^{(1)}\delta_2^{(m)} < \delta_2^{(1)}\delta_1^{(2)}$. From the ordering of the beliefs $\delta^{(i)}$, then $\delta_2^{(1)}\delta_1^{(2)} < \delta_2^{(1)}\delta_1^{(m)}$. Therefore, $\Delta < 0$.

The term in brackets defining \bar{g}_1 in equation (15) is strictly negative because $m \in K$. Thus, $\bar{g}_1 > 0$. Finally, using the condition of Case II and the order in $\delta^{(i)}$, then $\lambda_2/\lambda_1 > \delta_1^{(1)}/\delta_1^{(2)} > \delta_1^{(1)}/\delta_1^{(m)}$. Thus, $\bar{g}_2 > 0$.

For each $m \in K$, let $P^m = (x_m, y_m)$ be the solution of system (14). Define \bar{m} as the index corresponding to the largest x_m . There might be more than one \bar{m} , and if so, we pick any arbitrary \bar{m} such that $x_{\bar{m}} = \max\{x_m \mid m \in K\}$.

Claim: The contributions profile $g_1^{(\bar{m})} = x_{\bar{m}}/\beta_1^{(\bar{m})}$, $g_1^{(k)} = 0$ for every $k \neq \bar{m}$, and $g_2^{(1)} = y_{\bar{m}}/\beta_2^{(1)}$ satisfies conditions (12) and (13) for all $i \geq 2$; that is, this profile is an equilibrium.

The contributions given above generate the average contributions $\bar{g}_1 = x_{\bar{m}}$ and $\bar{g}_2 = y_{\bar{m}}$. Because $P^{\bar{m}}$ comes from system (14), then condition (12) is satisfied with "=", and condition (13) holds with "=" when $i = \bar{m}$.

Now, pick $m \in K - \{\bar{m}\}$. Because $P^{\bar{m}}$ satisfies the first equation of system (14), then:

$$\delta_1^{(m)}x_{\bar{m}} + \delta_2^{(m)}y_{\bar{m}} = \delta_1^{(m)}x_{\bar{m}} + \delta_2^{(m)} \left[\frac{\lambda_2}{\delta_2^{(1)}N} - \frac{\delta_1^{(1)}}{\delta_2^{(1)}}x_{\bar{m}} \right] = \left[\delta_1^{(m)} - \frac{\delta_2^{(m)}\delta_1^{(1)}}{\delta_2^{(1)}} \right] x_{\bar{m}} + \frac{\delta_2^{(m)}\lambda_2}{\delta_2^{(1)}N}.$$

The last bracket of the equalities above has the same signal of $\delta_1^{(m)}\delta_2^{(1)} - \delta_2^{(m)}\delta_1^{(1)} = -\Delta > 0$.

Using this and the definition of $x_{\bar{m}}$, then:

$$\delta_1^{(m)}x_{\bar{m}} + \delta_2^{(m)}y_{\bar{m}} \geq \left[\delta_1^{(m)} - \frac{\delta_2^{(m)}\delta_1^{(1)}}{\delta_2^{(1)}} \right] x_m + \frac{\delta_2^{(m)}\lambda_2}{\delta_2^{(1)}N} = \delta_1^{(m)}x_m + \delta_2^{(m)}y_m = \frac{\lambda_1}{N}.$$

The last two equalities are true because P^m is solution of system (14). Therefore, condition (13) is satisfied with "≥", for every $i \in K - \{\bar{m}\}$.

Finally, pick any $i \notin K$. Then, $\lambda_2/\lambda_1 \geq \delta_2^{(1)}/\delta_2^{(i)}$, or equivalently, $\delta_2^{(i)}N/\lambda_1 \geq \delta_2^{(1)}N/\lambda_2$. The condition of Case II and the order of the values $\delta^{(i)}$ imply that:

$$\frac{\lambda_2}{\lambda_1} > \frac{\delta_1^{(1)}}{\delta_1^{(2)}} \geq \frac{\delta_1^{(1)}}{\delta_1^{(i)}}.$$

Then, $\delta_1^{(i)}N/\lambda_1 > \delta_1^{(1)}N/\lambda_2$. Therefore:

$$\frac{\delta_1^{(i)}N}{\lambda_1}x_{\bar{m}} + \frac{\delta_2^{(i)}N}{\lambda_1}y_{\bar{m}} > \frac{\delta_1^{(1)}N}{\lambda_2}x_{\bar{m}} + \frac{\delta_2^{(1)}N}{\lambda_2}y_{\bar{m}} = 1.$$

The last equation holds because $P^{\bar{m}}$ is the solution of the first equation of system (14). Therefore, condition (13) holds with " $>$ " for every $i \notin K$. ■

Proof. (Proposition 4) Define $f(i) = \delta_1^{(i)}(\bar{g}_1 - \bar{g}_2) + \bar{g}_2$, for each $i \in I$. Then, the first order conditions characterizing the equilibrium become:

$$f(1) \geq \frac{\lambda_2}{N}; \quad \text{with "=" if } g_2^{(1)} > 0; \quad (16)$$

and, for every $i = 2, \dots, |I|$:

$$f(i) \geq \frac{\lambda_1}{N}; \quad \text{with "=" if } g_1^{(i)} > 0. \quad (17)$$

Claim I: $\bar{g}_1 > \bar{g}_2$.

If this claim is not true (namely $\bar{g}_1 \leq \bar{g}_2$), the function $f(i)$ would be a weakly decreasing function. From conditions (16) and (17), then $f(1) > \lambda_2/N$. Thus, $g_2^{(1)} = 0$ and $\bar{g}_2 = 0 = \bar{g}_1$, which is a contradiction.

The claim above implies that $f(i)$ is a strictly increasing function.

Claim II: $f(2) = \lambda_1/N$.

If $f(2) > \lambda_1/N$, then $f(i) > \lambda_1/N$, for all $i \geq 2$. Hence, $g_1^{(i)} = 0$ and $\bar{g}_1 = 0$, which means that (by Claim I) $\bar{g}_2 < 0$, but this is a contradiction.

Therefore, $f(i) > \lambda_1/N$ (because $f(i)$ is a strictly increasing function), and so, $g_1^{(i)} = 0$, for all $i \geq 3$. Thus, equations (16) and (17) have been reduced to:

$$f(1) \geq \frac{\lambda_2}{N}; \quad \text{with "=" if } g_2^{(1)} > 0,$$

and

$$f(2) = \frac{\lambda_1}{N}. \quad (18)$$

Part (i). Suppose $f(1) > \lambda_2/N$. This implies $g_2^{(1)} = 0$ (or $\bar{g}_2 = 0$) and substituting in (18), we find $g_1^{(2)} = \lambda_1/(\beta_1^{(2)}\delta_1^{(2)}N)$. Substituting in the condition of this case, $\delta_1^{(1)}\lambda_1/(\delta_1^{(2)}N) > \lambda_2/N$, which is the parameter condition in this part.

Part (ii). Suppose $f(1) = \lambda_2/N$. Let $\Delta = \delta_1^{(2)} - \delta_1^{(1)}$. By equation (18):

$$\begin{aligned} \bar{g}_1 &= \frac{\delta_2^{(2)}\lambda_1}{N\Delta} \left[-\frac{\lambda_2}{\lambda_1} + \frac{\delta_2^{(1)}}{\delta_2^{(2)}} \right], \\ \bar{g}_2 &= \frac{\delta_1^{(2)}\lambda_1}{N\Delta} \left[\frac{\lambda_2}{\lambda_1} - \frac{\delta_1^{(1)}}{\delta_1^{(2)}} \right]. \end{aligned}$$

The non-negativity condition $\bar{g}_2 \geq 0$ corresponds to the parameter condition in this part of the proposition, and it implies that $\lambda_2(1 - \delta_2^{(2)}) \geq \lambda_1(1 - \delta_2^{(1)})$ and $\lambda_1\delta_2^{(1)} - \lambda_2\delta_2^{(2)} \geq \lambda_1 - \lambda_2 > 0$. Hence, $\lambda_2/\lambda_1 < \delta_2^{(1)}/\delta_2^{(2)}$, which implies that $\bar{g}_1 > 0$. Computing $g_1^{(2)} = \bar{g}_1/\beta_1^{(2)}$ and $g_2^{(1)} = \bar{g}_2/\beta_2^{(1)}$, we obtain the result for this part.

Part (iii). The inequality $\beta_1^{(2)}g_1^{(2)} > \beta_2^{(1)}g_2^{(1)}$ is trivial in the case of Part (i). In the case of Part (ii), $\beta_1^{(2)}g_1^{(2)} > \beta_2^{(1)}g_2^{(1)}$ is equivalent to $\lambda_1 > \lambda_2$, which holds by assumption.

■

Proof. (Corollary 2):

Part (i). This result comes from the computation of $G^{**} = \sum_{(t,i) \in C} \beta_t^{(i)}\delta_t^*g_t^{(i)}N = \beta_1^{(2)}\delta_1^*g_1^{(2)}N = \delta_1^*\lambda_1/\delta_1^{(2)}$, where the values $g_t^{(i)}$ are the ones in the corresponding case of Proposition 4. Because $\delta_1^{(1)} < \delta_1^{(2)}$, then:

$$G^{**} = \frac{\delta_1^*}{\delta_1^{(2)}}\lambda_1 < \frac{\delta_1^*}{\delta_1^{(1)}}\lambda_1 = G^*.$$

Part (ii). In this case:

$$G^{**} = \sum_{(t,i) \in C} \beta_t^{(i)}\delta_t^*g_t^{(i)}N = \beta_1^{(2)}\delta_1^*g_1^{(2)}N + \beta_2^{(1)}\delta_2^*g_2^{(1)}N = \delta_1^* \frac{\delta_2^{(1)}\lambda_1 - \delta_2^{(2)}\lambda_2}{(\delta_1^{(2)} - \delta_1^{(1)})} + \delta_2^* \frac{\delta_1^{(2)}\lambda_2 - \delta_1^{(1)}\lambda_1}{(\delta_1^{(2)} - \delta_1^{(1)})}.$$

Substituting $\delta_2^* = 1 - \delta_1^*$ and grouping the coefficients of λ_1 and λ_2 , equation (9) follows.

Now, suppose that $\delta_1^{(1)} < \delta_1^*$. With some algebra:

$$G^{**} = \lambda_1 + (\lambda_1 - \lambda_2) \left(\frac{\delta_1^* - \delta_1^{(2)}}{\delta_1^{(2)} - \delta_1^{(1)}} \right). \quad (19)$$

Because $\lambda_1 > \lambda_2$ and $\delta_1^{(1)} < \delta_1^*$, then G^{**} is a decreasing function of $\delta_1^{(2)}$, for every $\delta_1^{(2)} \geq \delta_1^{(1)} \lambda_1 / \lambda_2$, and this inequality holds by the hypothesis of Part (ii). The maximum of G^{**} occurs exactly at $\delta_1^{(2)} = \delta_1^{(1)} \lambda_1 / \lambda_2$, and at this point (substituting this value of $\delta_1^{(2)}$ in equation (19), after some algebra):

$$\max_{\delta_1^{(2)} \geq \delta_1^{(1)} \lambda_1 / \lambda_2} G^{**} = \frac{\delta_1^*}{\delta_1^{(1)}} \lambda_2 < \frac{\delta_1^*}{\delta_1^{(1)}} \lambda_1 = G^*.$$

Part (iii). For Case (i), $G^{**} = \delta_1^* \lambda_1 / \delta_1^{(2)} > \lambda_1$ if and only if $\delta_1^{(2)} < \delta_1^*$.

For Case (ii), $G^{**} > \lambda_1$ if and only if

$$\left(\frac{\delta_1^* - \delta_1^{(1)}}{\delta_1^{(2)} - \delta_1^{(1)}} \right) \lambda_1 + \left(\frac{\delta_1^{(2)} - \delta_1^*}{\delta_1^{(2)} - \delta_1^{(1)}} \right) \lambda_2 > \lambda_1.$$

This inequality holds if and only if

$$\left(\frac{\delta_1^{(2)} - \delta_1^*}{\delta_1^{(2)} - \delta_1^{(1)}} \right) \lambda_2 > \left(\frac{\delta_1^{(2)} - \delta_1^*}{\delta_1^{(2)} - \delta_1^{(1)}} \right) \lambda_1,$$

or, equivalently, $(\lambda_1 - \lambda_2) \delta_1^* > (\lambda_1 - \lambda_2) \delta_1^{(2)}$ (because $\delta_1^{(2)} - \delta_1^{(1)} > 0$). This condition is satisfied if and only if $\delta_1^{(2)} < \delta_1^*$ because $\lambda_1 - \lambda_2 > 0$. ■

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